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THE UNTENABILITY OF LUCE'S PRINCIPLE

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In the course of his instructive researches in the mathematics of psychophysical scaling, R. Duncan Luce (1959a; 1959b, p. 29) has recently proposed a methodological principle which, if true, would have revolutionary import not merely for psychophysics, but for empirical and theoretical research in all branches of science. Moreover, while Luce's Principle has met with something less than enthusiastic acceptance in all quarters (cf. Luce, 1959b, p. 90; Mosteller, 1958, p. 287), others have begun to draw upon it as a basis for their own work (cf. Galanter & Messick, 1961). Such applications are decidedly premature, however, for the Principle's methodological status is dubious at best. As is inevitable for a provocatively novel hypothesis, certain ambiguities and unresolved difficulties reside in Luce's initial account, and when these are made explicit, it becomes highly problematic whether an interpretation can be found for the Principle which escapes being either vacuously true or empirically false.

Operating within the context of Stevens' theory of scale-types (e.g., Stevens, 1951),[1] Luce has proposed that the class of transformations defined by a scale-type should impose limitations on the mathematical form of the functional relations in which a variable of that type can participate. Specifically, it is postulated (Luce, 1959b) that:

> A substantive theory [i.e., an assumed law] relating two or more variables and the measurement theories [i.e., assumptions about the scale-types which are appropriate] for these variables should be such that:
>
> 1. (*Consistency of substantive and measurement theories*) Admissible transformations [i.e., transformations belonging to the class defined by the scale-type] of one or more of the independent variables shall lead, via the substantive theory, only to admissible transformations of the dependent variables.
>
> 2. (*Invariance of the substantive theory*) Except for the numerical values of parameters that reflect the effect on the dependent variables of admissible transformations of the independent variables, the mathematical structure of the substantive theory shall be independent of admissible transformations of the independent variables (p. 85).

Before putting his Principle to work, Luce comments that "one can hardly question the consistency part of the principle," but adds that "the invariance part is more subtle and controversial." I shall propose, to the contrary, that if this verbal statement of the Principle is taken at face value, it is the "consistency" part which is strong and correspondingly untenable, whereas the "invariance" part is a weakened version of the former which need be no more than a mathematical truism.

What can be meant when, in the consistency part of his Principle, Luce speaks of a transformation of an independent variable as *leading to*, via the substantive theory, a transformation of the dependent variable? The wording suggests that the transformation of the one somehow compels or induces a transformation of the other, an interpretation which is strengthened when, in subsequent illustration, Luce refers to a transformation being "effected on" the dependent variable. This cannot be a mathematical compulsion, however, for there is nothing about a lawful relationship which requires us to rescale one variable when transforming another. Thus given the law, "The area of a circle in square inches is π times its squared radius in inches," if I elect to scale length in feet rather than inches, my law simply becomes "The area of a circle in square inches is 144π times its squared radius

[1] I should also like to voice my reservations about a number of the conclusions of this theory. However, the substance of these doubts must await another occasion.

in feet." (If in this example the rescaling of length is felt—as it should not be—to require a corresponding rescaling of area, we could instead take, e.g., the relation between a person's height and his age.) The apparent meaning of the "consistency" stricture comes clearer upon examination of Luce's proffered illustration:

> Suppose it is claimed that two ratio scales are related by a logarithmic law. An admissible transformation of the independent variable x is multiplication by a positive constant k, i.e., a change of unit. However, the fact that $\log kx = \log k + \log x$ means that an inadmissible transformation, namely, a change of zero, is effected on the dependent variable. *Hence the logarithm fails to meet the consistency requirement* [italics added] (1959b, p. 85).

So worded, the example does not make explicit what happens to the dependent variable; however, let y be a function of x by the relation $y = \log x$, and let x' ($=_{\text{def}} kx$) be the transformed scale of the independent variable. Then the cited mathematical fact about $\log kx$ implies that $\log x' = \log k + \log x$, or $y + \log k = \log x'$; so if we wish to find a rescaling, y', of the dependent variable which stands in the same relation to x' as that in which y stands to x, it has to be $y' = y + \log k$, which is inadmissible (says Stevens) for a ratio scale. As demonstrated by this example, therefore, what the consistency injunction entails is that if an admissible transformation is made of the independent variable, then there must be an admissible rescaling of the dependent variable which leaves the mathematical expression of the law invariant—i.e., if $y = \phi x$, and x' is an admissible rescaling of x, then there is an admissible rescaling, y', of y, such that $y' = \phi x'$. Other than this, it is difficult to see what sense could be made of the notion that transformation of the one variable "effects" a transformation of the other.

Before defending this interpretation against another which also needs to be considered, it is useful to state explicitly just what does happen to a lawful relationship, $y = \phi x$, under transformation of its variables.[2] Let T and U be identity preserving (i.e., one-one) transformations, admissible or otherwise, applicable to x and y, respectively. Then if x is transformed into x' by T and y is transformed into y' by U, we have $x' = Tx$ and $y' = Uy$, or $x = T^{-1}x'$ and $y = U^{-1}y'$. Hence the substantive theory

$$y = \phi x$$

rewritten in terms of x' and y', becomes

$$U^{-1}y' = \phi T^{-1}x'$$

or

$$y' = U\phi T^{-1}x' \qquad [1]$$

Moreover, if it is *also* the case that $y' = \phi x'$, it follows from Equation 1 that $\phi x' = U\phi T^{-1}x'$, or $\phi Tx = U\phi x$. Let T_a be an *admissible* transformation of x, and U_a, similarly, an admissible transformation of y. Then the consistency claim of Luce's Principle apparently requires that if $y = \phi x$, the function ϕ must be such that for any admissible transformation T_a, there corresponds an admissible transformation U_a such that for all values of x,

$$\phi T_a x = U_a \phi x \qquad [2]$$

(It is easily seen that if the inverse of ϕ exists, this condition on ϕ is satisfied if and only if $\phi T_a \phi^{-1}$ is an admissible transformation of y.)

It has been proposed to me that what Luce intended the consistency part of his Principle to assert is only that if an admissible transformation, $x' = T_a x$, is made of the independent variable in $y = \phi x$, then there must exist some relationship ϕ' and an admissible transformation, $y' = U_a y$, of the dependent variable such that $y' = \phi' x'$. Then to derive Equation 2, the invariance part of the Principle must be construed to stipulate that U_a can always be chosen to set $\phi' = \phi$. But if this is all that the consistency claim were to imply, then it would have no restrictive force what-

[2] Since x may be construed as a vector, the analysis which follows applies to multivariate relationships as well as to laws involving only one independent variable.

soever—we can always take $\phi' = \phi T^{-1}$, in which case y requires no transformation at all—and hence a case could never arise, contrary to Luce's example cited earlier, in which the consistency requirement is violated. Moreover, as the invariance part of the Principle is worded, it is not strong enough to authorize Equation 2 because it insists not on *strict* invariance of ϕ under admissible transformations of the variables, but only on an invariance of some *structure* of ϕ. But the mathematical form (i.e., structure) of a function is hopelessly nonunique, and unless additional, as yet unspecified, conditions are imposed on this to-be-invariant form, we may take it to be a form common to any function of the class $\{U_a \phi T_a^{-1}\}$—which then leaves the invariance condition without any bite.

In any event, that Condition 2 is, in fact, the mathematical nugget which Luce (1959b) extracts from the ore of his Principle is demonstrated by his Table 1. Each entry in the "Functional Equation" column may be seen to be an instance of Equation 2. In view of the stiflingly strong consequences (e.g., Luce, 1959a, Table 2) of Luce's Principle, what can be said to justify it? Despite the undeniable mathematical charm of Equation 2, it seems to me that the answer must be an uncompromising "nothing whatsoever." It will be noticed that Luce himself does not actually present any argument to support his Principle, but in effect merely invites the assent of the reader's intuition. But intuition may well be gulled by the ambiguities in the verbal statement of the Principle into acceding to an innocuous reading of it which then imperceptibly slides into a not-so-innocuous version. This is especially likely for the consistency part of the Principle, which Luce insists can hardly be questioned even while at the same time holding it to imply at least *some* restriction on the form of a law. (But it has already been argued that the consistency claim is either vacuous or carries the full burden of Equation 2 by itself.) In any case, in my own methodological research I have seen too many initially self-evident theses wither away under careful probing to have much confidence in methodological intuition.

Pending cognitive support for Luce's Principle, then, how well does it stand the test of applicability to established laws? Well, as Luce himself points out, it is not difficult to find laws (see example below) which do *not* conform to it. Luce's comments on this faux pas carry the discussion into a rather different problem of vast proportions, but before pursuing his lead, we may pause to raise an eyebrow at the methodological status of a "principle" which is not binding. To be sure, the verbal statement of Luce's Principle does not say that all substantive laws *do* satisfy the Principle, but only that they *should* do so. It is most unlikely, however, that what was intended by this wording is merely that conformity to the Principle is *desirable*. Science is concerned with what is, not with what ought to be, and if the Principle is to have any scientific significance, it must be that Condition 2 says something about the forms which laws are *in fact* able to assume, and in light of which we may limit the forms to be considered when speculating about laws hitherto undiscovered. But if there are perfectly good laws which violate Equation 2, then it is difficult to see how the Principle could ever by itself impugn the legitimacy of any assumption about the form of a problematic law.

If Luce's Principle is to have any methodological force at all, therefore, there must exist additional criteria by which we can discriminate between laws which fall under the Principle's sovereignty and those which do not. Luce (1959b) has tentatively identified such a criterion: "All physical examples which have been suggested to me as counterexamples . . . have a common form: the independent variable is a ratio scale, but it enters the equation in a dimensionless fashion" (pp. 90 ff.). That is, the law can be put into a form in which the independent variable, x, is multiplied times a parameter, c, whose numerical value is determined by the choice of scale for x, so that any change in unit for x may be absorbed into c without disturbing the rest of the law—i.e., the law can be written in form $y = \phi_c(x) = \psi(cx)$, which under transformation $x' = kx$, goes over into $y = \phi_{c'}(x')$, where $c' = c/k$. Luce suggests that we can avoid violation of Condition 2 in such cases by taking the "dimensionless" variable v $(=_{\text{def}}\, cx)$ as our independent variable; then the law becomes $y = \psi(v)$, in which v has no admissible transformations and there are hence no restrictions imposed on ψ by Condition 2. Thus a more subtle version of Luce's Principle emerges: What is now postulated is not that Condition 2 is inviolable, but that the laws of nature are always such as to be *capable* of an expression which satisfies Equation 2.

Clearly crucial to this revised formulation of Luce's Principle, however, is the concept of a dimensionless variable. For if the method for satisfying Equation 2 in prima facie embarrassing cases is to search out a dimensionless expression of the independent variables, the restrictive force (if any) of the Principle cannot be comprehended until we understand what is involved in such a construction. So pregnant a topic as this amply merits discussion and controversy, which I will do my bit to provoke by injecting some irritants which, hopefully, may stimulate Luce and perhaps others to further production of methodological pearls.

It is my considered opinion that to render a variable dimensionless in the fashion brought to our attention by Luce amounts to nothing but a more or less arbitrary selection of one of the admissible scalings of that variable, and then working up an "absolute" interpretation for that scale. I contend that such an interpretation can be found for at least one admissible transformation of any variable, no matter what its scale type (in fact, I would argue that *every* scale can have an absolute interpretation forced upon it), and hence that there is no law on whose mathematical form the revised version of Luce's Principle could impose restrictions. To defend this thesis in full generality is impractical here, so I shall confine my remarks to ratio scales.

An example cited by Luce of a law which flouts Condition 2, but which can be brought to heel by dedimensionalizing it, is the law of radioactive decay. If q is the quantity, say in grams, of a certain radioactive substance S at any time t, say the number of seconds after the beginning of the Twentieth century, then

$$q = ae^{-bt} \qquad [3]$$

where a and b are empirical constants. So formulated, the dependent variable (q) is a ratio scale while the independent variable (t) is an interval scale, which apparently contravenes not only Condition 2 (cf. Luce, 1959b, Theorem 4), but also Luce's suggestion that the only violations of Equation 2 are ones in which the independent variables are ratio scales. However, it is debatable whether generalizations such as Equation 3, in which a time coordinate assumes the status of an independent variable, should be regarded as genuine laws (cf. Rozeboom, 1961, pp. 356f.). If t_0 is an arbitrary point in time on the t scale, and q_0 is the quantity (on the q scale) of S at time t_0, then Equation 3 may be rewritten as

$$q = q_0e^{-b(t-t_0)}$$

or letting $d =_{\text{def}} t - t_0$,

$$q = q_0e^{-bd} \qquad [4]$$

Equation 4, in which all the variables are ratio scales, relates the quantity (in grams) of substance S at the end of a given time interval jointly to the interval's duration (in seconds) and S's quantity (in grams) at the interval's beginning. We can also express Equation 4 as a law with only one independent variable by dividing both sides by q_0 and setting $p =_{\text{def}} q/q_0$. We then have

$$p = e^{-bd}, \qquad [5]$$

which shows what proportion of S at a given moment still remains d seconds later, and in which the dependent variable, p, is an absolute, or dimensionless, scale.

Neither Equation 4 nor Equation 5

yet satisfy Condition 2,[3] but if we now ask how long a "half-life," L_h, substance S has—i.e., L_h is the value of d for which $p = .5$ in Equation 5—we find that $L_h = (\ln 2)/b$, and hence that

$$q = (\tfrac{1}{2})^{d/L_h} \qquad [6]$$

Then letting d_h be the ratio of d to L_h—that is, if the time duration variable is rescaled with the half-life of S as the unit of measurement—we finally obtain

$$q = (\tfrac{1}{2})^{d_h} \qquad [7]$$

in which the independent variable is an absolute scale with no admissible transformations, and which is hence immune to any threat from Condition 2. This, then, is the sort of procedure which, suggests Luce, may be used to placate Condition 2 when an empirical law involving ratio scales (e.g., Equation 4) appears to disregard it.

It should be appreciated, however, that to describe a variable such as d_h in Equation 7 as dimensionless is somewhat misleading. The scale is just as much dependent upon a unit of measurement as it ever was; only now one particular unit has been singled out as a *preferred* unit, which determines an absolute scale of the variable in the sense (and merely in this sense) that there is one and only one scale which expresses each value of the variable as a multiple of this preferred unit. This can obviously be done whenever a preferred unit of measurement has been selected, and hence *any* ratio scale law, no matter what its mathematical structure, can in this fashion be put into a form which is compatible with Equation 2.

There is still one more gambit by which some restrictive force might conceivably be salvaged for Luce's Principle: If a distinction could be drawn between *natural* units of measurement on one hand, and merely conventional ones on the other, the definition of dimensionless scales could be limited to variables for which a natural unit is available. While it is my belief that "naturalness" in this sense is purely a psychological effect with little or no methodological significance, I will not argue the point here. Instead, I merely invite anyone who feels otherwise to state what unit is the "natural" one for dedimensionalizing the time-duration variable in the law of radioactive decay. There is nothing special about the half-life interval. For any proportion r, we could just as well take substance S's r-life, L_r, for our standard (i.e., L_r is the time required for S to decay to the proportion r of its initial amount), and write our law as

$$p = r^{d/L_r} \qquad [6']$$

or

$$p = r^{d_r}. \qquad [7']$$

where d_r is the L_r-scale of duration. Preference for $r = .5$ is wholly a matter of psychological convenience.

SUMMARY

The *strong* version of Luce's Principle, namely, that all laws must conform to Condition 2, is not only unsupported by reason, but is refuted by counterexamples. On the other hand, a modified version of the Principle which demands of a law only that it be equivalent to one which satisfies Equation 2 (or, somewhat less generally, that a law is exempt from Equation 2 only if its independent variables can be dedimensionalized) is vacuous insomuch as any law, no matter how seriously at odds with Equation 2 in its original form, can be coaxed into compliance with the weakened requirement. Until further clarifications and supports for Luce's Principle are forthcoming, then, it would be questionable strategy to limit speculations about possible laws to mathematical forms approved by Equation 2.

[3] Equation 4 violates Luce's Theorem 1 (1959b), while Equation 5 is especially interesting in that since its inverse exists and its dependent variable has no admissible transformations (other than the Identity transformation), Condition 2 requires that the same be true of its independent variable.

REFERENCES

GALANTER, E., & MESSICK, S. The relation between category and magnitude scales of loudness. *Psychol. Rev.*, 1961, **68**, 363–372.

LUCE, R. D. *Individual choice behavior: A theoretical analysis.* New York: Wiley, 1959. (a)

LUCE, R. D. On the possible psychophysical laws. *Psychol. Rev.*, 1959, **66**, 81–95. (b)

MOSTELLER, F. The mystery of the missing corpus. *Psychometrika*, 1958, **23**, 279–289.

ROZEBOOM, W. W. Ontological induction and the logical typology of scientific variables. *Phil. Sci.*, 1961, **28**, 337–377.

STEVENS, S. S. Mathematics, measurement and psychophysics. In S. S. Stevens (Ed.), *Handbook of experimental psychology.* New York: Wiley, 1951. Pp. 1–49.

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