CHAPTER 2. THE MEDIATION STRUCTURE OF MULTIVARIATE CAUSALITY

Although Chapter 1 sets out the main conceptual framework within which traditional notions of multivariate causality can best be reconstructed, we have scarcely begun to detail the theory of causal structure required to make sense of our intuitive interpretations of computed data parameters. Insomuch as this theory's motivation arises one level removed from the immediate practicalities of data analysis, readers whose interests in MODA are primarily applied may prefer to skim this chapter only lightly or omit it altogether if it distracts from their comprehension of MODA's operational character. For once we posit a particular model of well-specified form to explain our observations at hand--the conventional point of departure in the literature on causal modeling--little remains but to work out solutions for this model's parameter estimates and to appraise their sampling reliability. Nevertheless, when we move beyond particular solutions to contemplate a diversity of models for the same data array, or to compare results from several different studies purportedly dealing with the same phenomena, and realize that the differences manifest there may be complementation as much as conflict, we can appreciate need for a deeper understanding of causal relations.

The objectives of this chapter are really quite limited. Most importantly, we want to get clear about what might be called "causal micro-structure," namely, the logic by which one variable x has causal import for another, y, relative to some particular choice of supplementary y-sources Z that conjoin x in determining y while doing so through the mediation of still other y-sources that can also, though need not, be included with $\langle x, Z \rangle$ in assessment of joint effects on y. Our primary goal here is to identify the conditions under which the composition of one causal regularity into another is itself a (mediated) causal regularity. And we shall arrive at the wanted composition principle through a digraph representation of mediation structure which explicates and generalizes the notion of "causal path" that has long been intuitive in the literature on linear structural models. From there, we turn to some rudiments of causal macro-structure, which seeks to identify structural connections among aggregates of variables that are molar counterparts of micro-structural relations. What we are mainly after here is just a way to talk about causal mediation and causal determination among tuples of variables as wholes in a way that preserves the essential partial-order and composition properties of micro-causality without requiring our formalisms to be explicit about the underlying micro-structure.

As already acknowledged, none of the material developed in this chapter is explicitly required for MODA's application to particular data arrays. But some such theory is needed to explain what we are talking about when using MODA or any other multivariate method to make inferences about causal parameters.

To ease into this chapter's technicalities, it may help to review some presumptions/stipulations about variables and causal order proclaimed in Chapter 1. Among those worth a reminder are: (1) All variables at issue are jointly distributed over some fixed population $\underline{P}$, and any regularities, causal or otherwise, that we hypothesize to govern these variables are likewise prima facie relative to this $\underline{P}$. Henceforth, however, explicit reference to population $\underline{P}$ will be totally elided throughout this chapter. (2) The causal-source relation on pairs of variables is transitive, irreflexive, and is defined by same-subject causal regularities (over $\underline{P}$). (3) All tuples of variables are finite with no within-tuple repetitions; i.e., the variables within any specified tuple $\underset{\wedge}{X}$ are all distinct, and if tuples $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$ have any variables in common, $\langle\underset{\wedge}{X},\underset{\wedge}{Y}\rangle$ is not the $\underset{\wedge}{X}$-sequence continued by the $\underset{\wedge}{Y}$-sequence but only what remains of this concatenation after repetitions are deleted from the right. And (4), when $\underset{\wedge}{Y}$ is a subtuple of $\underset{\wedge}{X}$, not only are all $\underset{\wedge}{Y}$-variables also in $\underset{\wedge}{X}$, their order in $\underset{\wedge}{Y}$ is also the same as in $\underset{\wedge}{X}$.

Treating ensembles of variables as tuples, rather than unordered sets, is mandated by certain formal needs. But it has the infelicitous consequence of requiring recognition of order distinctions even where these are an irrelevant distraction. Specifically, many of the things we want to say about a given tuple $\underset{\wedge}{X}$ are true of $\underset{\wedge}{X}$

simply by virtue of what variables are in $\underset{\wedge}{X}$, regardless of how they are ordered therein. In such cases, when we have predicated such-and-such of $\underset{\wedge}{X}$, it seems awkward and artificial to add that such-and-such also holds for any other tuple containing the same variables as $\underset{\wedge}{X}$; nevertheless that addendum is generally needed, insomuch as if $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$ comprise the same variables in different orders, we are conceiving of them as formally distinct entities, and indeed, the such-and-such that holds for $\underset{\wedge}{X}$ may not be literally true of $\underset{\wedge}{Y}$ unless adjusted to take the order difference into account. Even so, when $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$ differ only by a permutation, it is heuristic to think of them as identical for most purposes. So to preserve the order difference formally while encouraging us to ignore this as a difference in substance, let us say

> *Definition* 2.1. $\underset{\wedge}{X}$ is *essentially identical with* $\underset{\wedge}{Y}$, symbolized $\underset{\wedge}{X} \doteq \underset{\wedge}{Y}$, iff every variable in $\underset{\wedge}{X}$ is also in $\underset{\wedge}{Y}$ and conversely. That is, given that the variables in any tuple are all distinct, $\underset{\wedge}{X} \doteq \underset{\wedge}{Y}$ iff $\underset{\wedge}{X} = \rho(\underset{\wedge}{Y})$ for some permutation $\rho(\underset{\wedge}{Y})$ of $\underset{\wedge}{Y}$.

We shall frequently want to refer to the variables in one tuple that are *not* also in another. Although this could be compactly formalized by introducing a special symbol for tuple subtraction, it seems more mnemonic to say

> *Definition* 2.2. $\underset{\wedge}{X}$-not-$\underset{\wedge}{Y}$ is the subtuple of variables $\underset{\wedge}{X}$ constructed by deleting from $\underset{\wedge}{X}$ each variable therein that is also in $\underset{\wedge}{Y}$. If all $\underset{\wedge}{X}$-variables are also in $\underset{\wedge}{Y}$, we say that $\underset{\wedge}{X}$-not-$\underset{\wedge}{Y}$ is the "null" tuple rather than that $\underset{\wedge}{X}$-not-$\underset{\wedge}{Y}$ does not exist.

Generally, we allow the order of variables in a tuple to be arbitrary. But it is occasionally convenient to exploit

> *Definition* 2.3. A tuple $\underset{\wedge}{X} = \langle \underset{\wedge}{x}_1, \ldots, \underset{\wedge}{x}_n \rangle$ of variables is *causally well-ordered* iff, for all $i, j = 1, \ldots, n$, $\underset{\wedge}{x}_i$ is a (causal) source of $\underset{\wedge}{x}_j$ only if $i < j$. *Theorem*: Every tuple $\underset{\wedge}{X}$ of variables has a permutation that is causally well-ordered.

The causal well-ordering theorem follows from our fundamental premise that the causal-source relation is a strict partial order, and is easily proved by induction on the number of variables in $\underset{\wedge}{X}$.

Finally, a distinction that will figure prominently in our forthcoming account of causal structure is

<u>Definition</u> 2.4. The (<u>causal</u>) <u>interior</u>, $\underline{I}(\underset{\wedge}{X})$, of a tuple $\underset{\wedge}{X}$ of variables is $\underset{\wedge}{X}$'s subtuple comprising just its variables that have a strictly complete source in $\underset{\wedge}{X}$. That is, $\underset{\wedge}{x}_j$ is in $\underline{I}(\underset{\wedge}{X})$ iff $\underset{\wedge}{x}_j$ is in $\underset{\wedge}{X}$ and some subtuple $\underset{\wedge}{X}_i$ of $\underset{\wedge}{X}$ is a strictly complete source of $\underset{\wedge}{x}_j$ under some nomically irreducible causal regularity $\underset{\wedge}{x}_j = \phi(\underset{\wedge}{X}_i)$. tr The (<u>causal</u>) <u>exterior</u>, $\underline{E}(\underset{\wedge}{X})$, of $\underset{\wedge}{X}$ comprises just the variables in $\underset{\wedge}{X}$ that do <u>not</u> have strictly complete sources in $\underset{\wedge}{X}$, i.e., $\underline{E}(\underset{\wedge}{X}) = \underset{\wedge}{X}$-not-$\underline{I}(\underset{\wedge}{X})$. Variable $\underset{\wedge}{x}_j$ is interior to $\underset{\wedge}{X}$ iff $\underset{\wedge}{x}_j$ is in $\underline{I}(\underset{\wedge}{X})$.

Obviously $\underline{E}(\underset{\wedge}{X}) \doteq \underline{E}(\underset{\wedge}{Y})$ and $\underline{I}(\underset{\wedge}{X}) \doteq \underline{I}(\underset{\wedge}{Y})$ whenever $\underset{\wedge}{X} \doteq \underset{\wedge}{Y}$. For compound tuples, we condense $\underline{E}(\langle \underset{\wedge}{X},\underset{\wedge}{Y}\rangle)$ to $\underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y})$ and $\underline{I}(\langle \underset{\wedge}{X},\underset{\wedge}{Y}\rangle)$ to $\underline{I}(\underset{\wedge}{X},\underset{\wedge}{Y})$. Later (p. 2.16f.), we shall prove that each variable in $\underline{I}(\underset{\wedge}{X})$ has a strictly complete source in $\underline{E}(\underset{\wedge}{X})$.

<u>Causal Micro-structure</u>.

So far as we have any reason to believe, whenever one variable causally affects another, it does so only indirectly through the mediation of others. Accordingly, the theory of causal regularity must above all be an account of mediated causality. In particular, we want this (<u>a</u>) to clarify what it is for the causal connection between two variables to be partially/wholly mediated by one or more others; (<u>b</u>) to envision how, in principle, a newly identified tuple $\underset{\wedge}{Z}$ of $\underset{\wedge}{Y}$-sources can be interlaced into previously established regularities under which variables $\underset{\wedge}{Y}$ are determined by variables $\underset{\wedge}{X}$; and (<u>c</u>) to spell out the conditions under which the composition of one causal regularity into another is itself a causal regularity. These matters prove to be rather more intricate than one might expect, and I am far from certain that the treatment now to be sketched is optimal. Nevertheless, it is a beginning.

The nature of partial mediation seems obvious: Variable x has some effect upon variable y through mediation by variable z just in case x is a source of z and z, in turn, is a source of y. But what is it for $x \rightarrow y$ to be <u>wholly</u> mediated by z or by a tuple Z? Or, when $x \rightarrow z \rightarrow y$, what demarks x's also having some influence upon y that is <u>not</u> mediated by z?

Consider the case where a tuple X of variables includes at least one, but possibly more than one, strictly complete source of variable y, where y may or may not be in X. Then more broadly, there exists a nonempty set $\{X_j\}$ of subtuples of X for each of which there is at least one transducer function ϕ_{ij} that maps each subject's score tuple on X_j into that subject's score on y. Let us momentarily call any such factual regularity $y = \phi_{ij}(X_j)$ a "binding" of y by X_j within X, regardless of whether it is strictly causal. (If y is in X, $y = y$ also counts as a binding of y within X.) For each binding $y = \phi_{ij}(X_j)$ of y by X_j, and every subtuple X_k of X that includes X_j, there is also at least one binding $y = \phi_{ik}(X_k)$ of y by X_k, most evidently but not in general exclusively the one for which $\phi_{ik}(X_k) = \phi_{ij}\sigma_{jk}(X_k)$ where σ_{jk} is a subtuple-selector function over tuples of appropriate order such that $X_j = \sigma_{jk}(X_k)$. (Expressed as a matrix-algebraic premultiplier, σ_{jk} is the matrix whose <u>hi</u>th element is 1 or 0 according to whether the <u>h</u>th variable in X_j is or is not the <u>i</u>th variable in X_k.) Whenever the function ϕ in a binding $y = \phi(X_j)$ can be decomposed as $\phi = \psi\sigma$ for some subset-selector function σ, it will be convenient to to say that the variables in X_j-not-$\sigma(X_j)$ have "null weight" or "zero weight" in $y = \phi(X_j)$, since if X'_j is the subtuple of X_j picked out by σ, i.e. $X'_j = \sigma(X_j)$, and the variables $X''_j =_{def} X_j$-not-X'_j not selected out of X_j by σ occur after X'_j in X_j, i.e. $X_j = \langle X'_j, X''_j \rangle$, then $\phi(X_j) = \psi\sigma(X_j)$ is equivalent to $\phi(X_j) = \psi(X'_j) + 0 \cdot X''_j$. By presumption, at least one of y's bindings $\{y = \phi_{ij}(X_j)\}$ by subtuples of X is a nomically irreducible causal regularity—but what can we say about the causal status of these other bindings? It will suffice to discuss the case $X_j = X$ and omit the subtuple subscript.

(Later, we shall define null weights to be a special case of zero weights.)

<u>Proximalities</u>.

Our prior postulation (p. 12) that the transducer of a causal regularity (see p. 1.21f.) is unique even when its input variables are not fully dispersed entails that when $\underset{\wedge}{X}$ is a strictly complete source of $\underset{\wedge}{y}$, just one function ϕ^* in the multiplicity of $\underset{\wedge}{y}$'s bindings by $\underset{\wedge}{X}$ is truly causal in the sense of characterizing how the variables in $\underset{\wedge}{X}$ work jointly to bring about $\underset{\wedge}{y}$. It seems entirely reasonable to posit more broadly that even when only a proper subtuple of $\underset{\wedge}{X}$ is a strictly complete source of $\underset{\wedge}{y}$, there is just one binding $\underset{\wedge}{y} = \phi^*(\underset{\wedge}{X})$ of $\underset{\wedge}{y}$ by $\underset{\wedge}{X}$ that tells how the variables in $\underset{\wedge}{X}$ causally determine $\underset{\wedge}{y}$ jointly, with some $\underset{\wedge}{X}$-variables given null weight by ϕ^* in $\phi^*(\underset{\wedge}{X})$ either because they are not sources of $\underset{\wedge}{y}$ at all (including $\underset{\wedge}{y}$ itself when $\underset{\wedge}{y}$ is in $\underset{\wedge}{X}$) or because, relative to the entirety of $\underset{\wedge}{X}$, they influence $\underset{\wedge}{y}$ only indirectly through their effects on other $\underset{\wedge}{y}$-sources in $\underset{\wedge}{X}$ and contribute nothing to $\underset{\wedge}{y}$ over and above the latter. Let us call this special binding of $\underset{\wedge}{y}$ by $\underset{\wedge}{X}$ an <u>inclusive causal regularity</u> whose transducer is ϕ^* and under which $\underset{\wedge}{X}$ is an <u>inclusively complete source</u> of $\underset{\wedge}{y}$. (We shall understand inclusive causal regularities, and inclusively complete sources, to subsume strict ones as a special case--i.e., a strict causal regularity is an inclusive one in which no input variable has null weight.) Whenever $\underset{\wedge}{y} = \phi(\underset{\wedge}{X})$ is an inclusive but not strict causal regularity, there must be at least one variable $\underset{\wedge}{x}_0$ in $\underset{\wedge}{X}$ that has null weight in $\phi(\underset{\wedge}{X})$ and which can be deleted from $\underset{\wedge}{y} = \phi(\underset{\wedge}{X})$ without degrading the reduced function's causal status--i.e., $\phi(\underset{\wedge}{X}) = \phi_0\sigma_0(\underset{\wedge}{X})$ in this case, where $\sigma_0(\underset{\wedge}{X}) = \underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0$ and $\underset{\wedge}{y} = \phi_0(\underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0)$ is an inclusive causal regularity under which $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0$ is an inclusively complete source of $\underset{\wedge}{y}$. Accordingly, deletion of null-weight variables can be iterated until the original inclusive causal regularity is reduced to a strict one whose input variables are just the ones in $\underset{\wedge}{X}$ that have effects on $\underset{\wedge}{y}$ unmediated by the others. We may call this special subtuple of $\underset{\wedge}{X}$ the "proximal" source of $\underset{\wedge}{y}$ in $\underset{\wedge}{X}$ and begin to characterize its causal role as follows:

<u>Causal-mediation Postulate</u> 1 [<u>CmP</u>-1]. For any tuple $\underset{\wedge}{X}$ of variables that is an inclusively complete source of some variable $\underset{\wedge}{y}$, i.e. of which some subtuple is a strictly complete source of $\underset{\wedge}{y}$, exactly one binding $\underset{\wedge}{y} = \phi(\underset{\wedge}{X})$ of $\underset{\wedge}{y}$ by $\underset{\wedge}{X}$ is

an inclusive causal regularity under which variables X determine y jointly; and there is exactly one subtuple X^* of X such that if σ^* is the subtuple-selector function that picks X^* out of X (i.e. $X^* = \sigma^*(X)$), the transducer of inclusive causal regularity $y = \phi(X)$ has composition $\phi = \phi^*\sigma^*$ where ϕ^* is the transducer of a strict causal regularity $y = \phi^*(X^*)$ under which X^* is a strictly complete source of y. By definition, this special subtuple X^* of X is the (<u>complete</u>) <u>proximal</u> <u>source</u> of y in X. If y has no strictly complete source in X, we shall say that the proximal source of y in X is <u>null</u>.

It is immediate from <u>CmP</u>-1 that X is a strictly complete source of y just in case X is an inclusively complete source of y within which X is also the proximal source of y.

Just as different subtuples of X can be inclusively or even strictly complete sources of y even though among these only one--y's proximal source in X--is causally immediate for y in X, so is there in general a corresponding multiplicity of causal regularities under which y is determined by its sources in X albeit all but one of these are derived by composition from others. To study these mediation relations, it proves most convenient to include output variable y in the tuple X among whose subtuples we find a diversity of complete y-sources. Then we can say

<u>Definition</u> 2.5. An inclusive (possibly strict) causal regularity $x_j = \phi(X_i)$ is <u>within</u> (or <u>in</u>) a tuple X of variables iff x_j is in $\underline{I}(X)$ and X_i is a subtuple of X. A causal regularity $x_j = \phi(X_i^*)$ is <u>proximal</u> in X iff it is within X and X_i^* is the proximal source of x_j in X.

Any causal regularity that is proximal in X is necessarily strict. Obviously, if variables $\langle x_j, X_i^* \rangle$ are all in X, X_i^* is a strictly complete source of x_j just in case X_i^* determines x_j under some causal regularity $x_j = \phi(X_i^*)$ that is proximal within at least one subtuple X' of X, notably $X' \doteq \langle x_j, X_i^* \rangle$.

It is manifest in the intuitive reasoning behind <u>CmP</u>-1 that the proximal source X_i^* of x_j in X should also be the proximal source of x_j in any subtuple of X that contains X_i^*. The same is not generally true when X is augmented rather than diminished, however: for if z is a variable that mediates between x_j and some x_k in X_i^*, our intuitions about mediation structure allow that the proximal source of x_j in

$\langle \underset{\wedge}{X}, \underset{\wedge}{z} \rangle$ may well include $\underset{\wedge}{z}$ instead of or--if $\underset{\wedge}{z}$ mediates only part of the $\underset{\wedge}{x}_k \to \underset{\wedge}{y}$ connection--in addition to $\underset{\wedge}{x}_k$. Indeed, intuition insists that the proximal source of $\underset{\wedge}{x}_j$ in $\langle \underset{\wedge}{X}, \underset{\wedge}{z} \rangle$ must include $\underset{\wedge}{z}$ if none of the other variables in $\underset{\wedge}{X}_i^*$ in turn mediates between $\underset{\wedge}{z}$ and $\underset{\wedge}{x}_j$. On the other hand, if $\underset{\wedge}{z}$ does <u>not</u> mediate between $\underset{\wedge}{x}_j$ and any other variable in $\underset{\wedge}{X}_i^*$, then $\underset{\wedge}{X}_i^*$ remains the proximal source of $\underset{\wedge}{y}$ in $\langle \underset{\wedge}{X}, \underset{\wedge}{z} \rangle$. Yet these are just two of many causal-structure principles that seem apodictic. We need to regiment these intuitions by expanding <u>CmP</u>-1 into a complete axiomatic foundation for them.

Consider an arbitrary tuple $\underset{\wedge}{X}$ of variables with an interestingly non-null causal interior $\underline{I}(\underset{\wedge}{X})$. Each variable $\underset{\wedge}{x}_j$ in $\underline{I}(\underset{\wedge}{X})$ by definition has a strictly complete source in $\underset{\wedge}{X}$; so by <u>CmP</u>-1, $\underset{\wedge}{x}_j$ has a (complete) proximal source $\underset{\wedge}{X}_i^*$ in $\underset{\wedge}{X}$. If we take note of which $\underset{\wedge}{X}$-variables are in the proximal sources of which others relative to $\underset{\wedge}{X}$, it is instructive to consider how these proximalities are altered relative to some minimally reduced subtuple $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$ of $\underset{\wedge}{X}$. A concept that proves to be remarkably powerful in thinking through this matter is

<u>Definition</u> 2.6. Variable $\underset{\wedge}{x}_i$ is a <u>direct</u> <u>source</u> of variable $\underset{\wedge}{x}_j$ within tuple $\underset{\wedge}{X}$ (i.e., relative to $\underset{\wedge}{X}$) iff $\underset{\wedge}{x}_j$ is interior to $\underset{\wedge}{X}$ and the proximal source of $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ includes $\underset{\wedge}{x}_i$.

Given <u>CmP</u>-1, a variable $\underset{\wedge}{x}_j$ is interior to $\underset{\wedge}{X}$ just in case it has a direct source within $\underset{\wedge}{X}$, whereas if $\underset{\wedge}{x}_j$ is in $\underset{\wedge}{X}$ but has no direct source within $\underset{\wedge}{X}$, $\underset{\wedge}{x}_j$ is in the exterior of $\underset{\wedge}{X}$. And the subtuple of $\underset{\wedge}{X}$ comprising just the variables that are direct sources of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$ is $\underset{\wedge}{x}_j$'s proximal source in $\underset{\wedge}{X}$. Consequently, we can represent which subtuples of $\underset{\wedge}{X}$ are proximal sources of which other $\underset{\wedge}{X}$-variables by a digraph whose nodes correspond to the variables in $\underset{\wedge}{X}$ and which includes an arrow from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ just in case $\underset{\wedge}{x}_i$ is a <u>direct</u> source of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$.

<u>Causal</u>-<u>mediation</u> <u>Postulate</u> 2 [<u>CmP</u>-2]. Let $\underset{\wedge}{x}_0$, $\underset{\wedge}{x}_i$, and $\underset{\wedge}{x}_j$ be any distinct variables in tuple $\underset{\wedge}{X}$. Then deletion of $\underset{\wedge}{x}_0$ from $\underset{\wedge}{X}$ affects the direct-source relation of $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ relative to $\underset{\wedge}{X}$ vs. $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$ as follows: (<u>a</u>) If $\underset{\wedge}{x}_0$ is not a direct source of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$, then $\underset{\wedge}{x}_i$ is a direct source of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$

if and only if $\underset{\wedge}{x}_i$ is a direct source of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$. (b) If $\underset{\wedge}{x}_0$ and $\underset{\wedge}{x}_i$ are both direct sources of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$, then $\underset{\wedge}{x}_i$ is a direct source of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$ if but only if $\underset{\wedge}{x}_0$ is interior to $\underset{\wedge}{X}$ (i.e., if $\underset{\wedge}{x}_0$ has a direct source of its own within $\underset{\wedge}{X}$ but not otherwise). (c) If $\underset{\wedge}{x}_0$ but not $\underset{\wedge}{x}_i$ is a direct source of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$, then $\underset{\wedge}{x}_i$ is a direct source of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$ just in case $\underset{\wedge}{x}_i$ is a direct source of $\underset{\wedge}{x}_0$ within $\underset{\wedge}{X}$.

CmP-2a is equivalent to saying that any proximal regularity within $\underset{\wedge}{X}$ is also a proximal regularity in any subtuple of $\underset{\wedge}{X}$ that contains the requisite variables--the cogency of which we have already observed in slightly different terms. CmP-2b recognizes that if $\underset{\wedge}{x}_j = \phi(\underset{\wedge}{X}^*_i)$ is a proximal regularity in $\underset{\wedge}{X}$, it cannot be so in $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$ if $\underset{\wedge}{x}_0$ is one of the variables in $\underset{\wedge}{X}^*_i$. And if $\underset{\wedge}{X}^*_i$ includes $\underset{\wedge}{x}_0$, $\underset{\wedge}{X}^*_i$-not-$\underset{\wedge}{x}_0$ is not a strictly complete source of $\underset{\wedge}{x}_j$ (since otherwise $\underset{\wedge}{x}_j = \phi(\underset{\wedge}{X}^*_i)$ would not be nomically irreducible); so either $\underset{\wedge}{x}_0$ has a complete source of its own in $\underset{\wedge}{X}$--in which case, replacing $\underset{\wedge}{x}_0$ in $\underset{\wedge}{X}^*_i$ by $\underset{\wedge}{x}_0$'s own proximal source in $\underset{\wedge}{X}$ gives a strictly complete source of $\underset{\wedge}{x}_j$ that is as causally close to $\underset{\wedge}{x}_j$ as we can get in $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$--or $\underset{\wedge}{x}_0$ is in $\underset{\wedge}{X}$'s exterior whence the sources of $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$ are insufficient to determine $\underset{\wedge}{x}_j$ fully. (Note that this argument for CmP-2b is not a proof, but only an exercising of intuitions that this postulate formalizes.) And CmP-2c explains how mediated causality becomes direct connection relative to a suitably frugal selection of the output variable's conjoint sources.

It is routine though somewhat tedious to show (the proof will be omitted here) that from any admissible structure of direct-source relations within a tuple $\underset{\wedge}{X}$, CmP-2 derives the same direct-source structure within ($\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_1$)-not-$\underset{\wedge}{x}_0$ as within ($\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$)-not-$\underset{\wedge}{x}_1$ for any two variables $\underset{\wedge}{x}_0$ and $\underset{\wedge}{x}_1$ in $\underset{\wedge}{X}$--as indeed it must if CmP-2 is to be coherent. Consequently, given the direct-source structure within any tuple $\underset{\wedge}{X}$, CmP-2 identifies a unique direct-source structure within any subtuple $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ of $\underset{\wedge}{X}$. And if $\underset{\wedge}{X}_0 = \langle \underset{\wedge}{X}_1, \underset{\wedge}{X}_2 \rangle$, the direct-source structure so derived first within $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_1$ and from there within ($\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_1$)-not-$\underset{\wedge}{X}_2$ is the same as within

$\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$. Conversely, if we are given the direct-source structure just within some subtuple $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ of $\underset{\wedge}{X}$, *CmP*-2 describes constraints on the direct-source structure within $\underset{\wedge}{X}$ imposed by the structure within $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$.

Case-by-case comparisons show that *CmP*-2 is equivalent to

Theorem 1. Let $\underset{\wedge}{x}_0$ be any variable in tuple $\underset{\wedge}{X}$, so that $\underset{\wedge}{x}_0$ is either in $\underline{I}(\underset{\wedge}{X})$ or in $\underline{E}(\underset{\wedge}{X})$ but not both. (*a*) Suppose that $\underset{\wedge}{x}_0$ is interior to $\underset{\wedge}{X}$. Then all variables other than $\underset{\wedge}{x}_0$ that are interior to $\underset{\wedge}{X}$ are also interior to $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$, and all variables in the exterior of $\underset{\wedge}{X}$ are also in the exterior of $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$. More specifically, for any variable $\underset{\wedge}{x}_j \neq \underset{\wedge}{x}_0$ in $\underline{I}(\underset{\wedge}{X})$, the proximal source of $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$ comprises just the variables other than $\underset{\wedge}{x}_0$ (if any) that are direct sources of $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ together with, if $\underset{\wedge}{x}_0$ is a direct source of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$, the variables that are direct sources of $\underset{\wedge}{x}_0$ within $\underset{\wedge}{X}$. (*Corollary*. If $\underset{\wedge}{X}_0$ is a subtuple of $\underline{I}(\underset{\wedge}{X})$, $\underline{E}(\underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0) = \underline{E}(\underset{\wedge}{X})$ and $\underline{I}(\underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0) = \underline{I}(\underset{\wedge}{X})\text{-not-}\underset{\wedge}{x}_0$.) (*b*) Alternatively, let $\underset{\wedge}{x}_0$ be in the exterior of $\underset{\wedge}{X}$. Then the interior of $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$ comprises just the variables in $\underline{I}(\underset{\wedge}{X})$ of which $\underset{\wedge}{x}_0$ is not a direct source within $\underset{\wedge}{X}$, so that $\underline{E}(\underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0)$ comprises all variables in $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$ that are either in $\underline{E}(\underset{\wedge}{X})$ or have $\underset{\wedge}{x}_0$ for a direct source in $\underset{\wedge}{X}$; and each variable in $\underline{I}(\underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0)$ has the same direct sources in $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$ as it has in $\underset{\wedge}{X}$. (*Corollary*. Statement (*b*) remains true if $\underset{\wedge}{x}_0$ is replaced by any subtuple $\underset{\wedge}{X}_0$ of $\underline{E}(\underset{\wedge}{X})$.)

Theorem 1 is easier to visualize in direct-source digraphs for $\underset{\wedge}{X}$ and $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$ than is *CmP*-2, and will be our main point of departure for subsequent theorems.

Causal Paths.

A variable that is the second term in one direct-source linkage within $\underset{\wedge}{X}$ can also be the first term of another. Iteration of this notion gives

Definition 2.7. A (*causal*) *path* (*of length* $\underline{m}$) in any tuple $\underset{\wedge}{X}$ of variables is any sequence $\underset{\wedge}{X}' = \langle \underset{\wedge}{x}'_1, \ldots, \underset{\wedge}{x}'_{m+1} \rangle$ of variables in $\underset{\wedge}{X}$ such that for each $\underline{k} = 1, \ldots, \underline{m}$, $\underset{\wedge}{x}'_k$ is a direct source of $\underset{\wedge}{x}'_{k+1}$ within $\underset{\wedge}{X}$. A path $\underset{\wedge}{X}'$ in $\underset{\wedge}{X}$ is *from* $\underset{\wedge}{x}_1$ iff $\underset{\wedge}{x}_1$ is the first variable in $\underset{\wedge}{X}'$, and is <u>to</u> $\underset{\wedge}{x}_j$ iff $\underset{\wedge}{x}_j$ is the last variable in $\underset{\wedge}{X}'$. A <u>total path</u> to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ is a path in $\underset{\wedge}{X}$ to $\underset{\wedge}{x}_j$ from some variable in $\underline{E}(\underset{\wedge}{X})$. If $\underset{\wedge}{X}' = \langle \underset{\wedge}{X}_a, \underset{\wedge}{X}_b \rangle$ is a path in $\underset{\wedge}{X}$ with $\underset{\wedge}{X}_a$ but not $\underset{\wedge}{X}_b$ possibly null, $\underset{\wedge}{X}_b$ is a <u>terminal segment</u> of $\underset{\wedge}{X}'$ with $\underset{\wedge}{X}_a$ the corresponding <u>initial segment</u> of $\underset{\wedge}{X}'$. A path $\underset{\wedge}{X}'$ in $\underset{\wedge}{X}$ <u>passes through</u> a tuple $\underset{\wedge}{X}_k$ of variables iff $\underset{\wedge}{X}'$ includes at least one variable in $\underset{\wedge}{X}_k$.

How these path concepts are represented in a direct-source digraph will be obvious.

Various consequences of this definition too immediate to formalize as theorems are: (1) For any path $\underset{\wedge}{X}'$ in $\underset{\wedge}{X}$, the variables in $\underset{\wedge}{X}'$ are all distinct (else the causal-source relation could not be a strict partial order); hence any path in $\underset{\wedge}{X}$ can be characterized as a tuple of variables without violating our convention that the variables in a tuple are all distinct. Moreover, if $\underset{\wedge}{X}$ is causally well-ordered, each path $\underset{\wedge}{X}'$ in $\underset{\wedge}{X}$ is a subtuple of $\underset{\wedge}{X}$, i.e. the sequence of variables in $\underset{\wedge}{X}'$ is the same as their order in $\underset{\wedge}{X}$. (2) If $\underset{\wedge}{Z} \doteq \underset{\wedge}{X}$, all paths in $\underset{\wedge}{X}$ are also paths in $\underset{\wedge}{Z}$. (3) Tuple $\underset{\wedge}{X}'$ is a path in $\underset{\wedge}{X}$ just in case each adjacent 2-tuple in $\underset{\wedge}{X}'$ is a length-1 path in $\underset{\wedge}{X}$. (4) All variables except possibly the first in any path in $\underset{\wedge}{X}$ are interior to $\underset{\wedge}{X}$, and there is a path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ just in case $\underset{\wedge}{x}_j$ is interior to $\underset{\wedge}{X}$. (5) If $\underset{\wedge}{X}_a$ and $\underset{\wedge}{X}_b$ are non-null, $\langle \underset{\wedge}{X}_a, \underset{\wedge}{X}_b \rangle$ is a path in $\underset{\wedge}{X}$ just in case $\underset{\wedge}{X}_a$ and $\underset{\wedge}{X}_b$ are paths in $\underset{\wedge}{X}$ with the last variable in $\underset{\wedge}{X}_a$ a direct source within $\underset{\wedge}{X}$ of the first variable in $\underset{\wedge}{X}_b$. (6) Each path $\underset{\wedge}{X}_{ij}$ from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ is the terminal segment of a total path $\langle \underset{\wedge}{X}_a, \underset{\wedge}{X}_{ij} \rangle$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ wherein $\underset{\wedge}{X}_a$ is null just in case $\underset{\wedge}{x}_i$ is in $\underline{E}(\underset{\wedge}{X})$. (7) Whenever $\underset{\wedge}{X}_{ij}$ is a path from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ of length greater than 1, $\underset{\wedge}{X}_{ij}$-not-$\underset{\wedge}{x}_i$ and $\underset{\wedge}{X}_{ij}$-not-$\underset{\wedge}{x}_j$ are also paths in $\underset{\wedge}{X}$. And (8) when $\langle \underset{\wedge}{X}_a, \underset{\wedge}{x}_k, \underset{\wedge}{X}_b \rangle$ is a path in $\underset{\wedge}{X}$, it is possible but not necessary that $\langle \underset{\wedge}{X}_a, \underset{\wedge}{X}_b \rangle$ is also a path in $\underset{\wedge}{X}$. (The latter obtains just in case the last variable in $\underset{\wedge}{X}_a$ is a direct source in $\underset{\wedge}{X}$ of the first variable in $\underset{\wedge}{X}_b$ as well as of $\underset{\wedge}{x}_k$.) Thus one path from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ can be a proper subtuple of another.

If $\underset{\wedge}{X}_0$ is a tuple of variables in $\underset{\wedge}{X}$, how does the path structure in $\underset{\wedge}{X}$'s subtuple $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ relate to the path structure in $\underset{\wedge}{X}$? This is best seen by starting with the special cases wherein $\underset{\wedge}{X}_0$ is restricted to variables either ($\underline{I}$) all in $\underline{I}(\underset{\wedge}{X})$ or ($\underline{E}$) all in $\underline{E}(\underset{\wedge}{X})$. And without essential loss of generality we can avoid certain nusiance complications by examining just total paths in $\underset{\wedge}{X}$ vs. $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ to variables interior to $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$.

For Case I, assume that $\underset{\wedge}{X}_0$ contains only variables interior to $\underset{\wedge}{X}$. Then from Th. 2a, by induction on the number of variables in $\underset{\wedge}{X}_0$, a 2-tuple $\langle \underset{\wedge}{x}_i, \underset{\wedge}{x}_j \rangle$ of variables in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ is a (length-1) path in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ just in case there is some possibly-null tuple $\underset{\wedge}{X}'_0$ of variables in $\underset{\wedge}{X}_0$ such that $\langle \underset{\wedge}{x}_i, \underset{\wedge}{X}'_0, \underset{\wedge}{x}_j \rangle$ is a path in $\underset{\wedge}{X}$. From there, together with the identity of $\underline{E}(\underset{\wedge}{X})$ with $\underline{E}(\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0)$ in this case (cf. Th. 1a), it is easy to see that any $\underset{\wedge}{X}'$ is a total path in $\underset{\wedge}{X}$ to some $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ only if $\underset{\wedge}{X}'$-not-$\underset{\wedge}{X}_0$ is a total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$, while $\underset{\wedge}{X}''$ is a total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ only if $\underset{\wedge}{X}'' = \underset{\wedge}{X}'$-not-$\underset{\wedge}{X}_0$ for some total path $\underset{\wedge}{X}'$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$.

For Case E, assume instead that $\underset{\wedge}{X}_0$ contains only variables in $\underset{\wedge}{X}$'s exterior. Then by Th. 1b, any total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ is also a path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ and is hence the terminal segment of some total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$. Conversely, let $\underset{\wedge}{X}'$ be any total path in $\underset{\wedge}{X}$ to some $\underset{\wedge}{x}_j$ interior to $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$. Although $\underset{\wedge}{X}'$-not-$\underset{\wedge}{X}_0$ differs from $\underset{\wedge}{X}'$ by deletion of at most the first variable in $\underset{\wedge}{X}'$ (since all subsequent variables in $\underset{\wedge}{X}'$ are in $\underline{I}(\underset{\wedge}{X})$ and hence not in $\underset{\wedge}{X}_0$), $\underset{\wedge}{X}'$-not-$\underset{\wedge}{X}_0$ need not be a total path, or even a path at all, in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ because some variables after the first in $\underset{\wedge}{X}'$-not-$\underset{\wedge}{X}_0$ may have some $\underset{\wedge}{X}_0$-variables as direct sources in $\underset{\wedge}{X}$ and hence (cf. Th. 2b) have no direct sources in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ at all. Even so, stipulation that $\underset{\wedge}{x}_j$ is interior to $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ with $\underset{\wedge}{X}'$ a total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ entails, from Th. 1b, that $\underset{\wedge}{X}'$ has some segmentation $\underset{\wedge}{X}' = \langle \underset{\wedge}{X}_a, \underset{\wedge}{X}_b \rangle$ wherein the first variable $\underset{\wedge}{x}_i$ in $\underset{\wedge}{X}_b$ but no other variable in $\underset{\wedge}{X}_b$ is in $\underline{E}(\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0)$--either because $\underset{\wedge}{x}_i$ is the rightmost variable in $\underset{\wedge}{X}'$ of which some $\underset{\wedge}{X}_0$-variable is a direct source within $\underset{\wedge}{X}$ or because, when no $\underset{\wedge}{X}_0$-variable is a direct source within $\underset{\wedge}{X}$ of any variable in $\underset{\wedge}{X}'$, $\underset{\wedge}{X}_b = \underset{\wedge}{X}'$ with $\underset{\wedge}{X}_a$ null--so that $\underset{\wedge}{X}_b$ is a total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$. Thus when all $\underset{\wedge}{X}_0$-variables are in $\underline{E}(\underset{\wedge}{X})$, $\underset{\wedge}{X}'$ is a total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ only if some terminal segment of $\underset{\wedge}{X}'$ is a total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$, while conversely, as already observed, $\underset{\wedge}{X}''$ is a total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ only if $\underset{\wedge}{X}''$ is the terminal segment of some total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$.

More generally, combining Cases $\underline{I}$ and $\underline{E}$, any tuple $\underset{\wedge}{X}_0$ of variables in $\underset{\wedge}{X}$ can be partitioned as $\underset{\wedge}{X}_0 \doteq \langle \underset{\wedge}{X}_1, \underset{\wedge}{X}_2 \rangle$ where $\underset{\wedge}{X}_1$ is some possibly-null subtuple of $\underline{I}(\underset{\wedge}{X})$, and $\underset{\wedge}{X}_2$ is some possibly-null subtuple of $\underline{E}(\underset{\wedge}{X})$ which is then also a subtuple of $\underline{E}(\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_1)$ (since $\underline{E}(\underset{\wedge}{X}) = \underline{E}(\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_1)$ by <u>Th.</u> 1<u>a</u>). By Case $\underline{I}$, a tuple $\underset{\wedge}{X}'$ of $\underset{\wedge}{X}$-variables is a total path in $\underset{\wedge}{X}$ to some $\underset{\wedge}{x}_j$ that is in $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_0$ and hence in $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_1$ only if $\underset{\wedge}{X}'\text{-not-}\underset{\wedge}{X}_1$ is a total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_1$, which in turn entails under Case $\underline{E}$ that some terminal segment of $(\underset{\wedge}{X}'\text{-not-}\underset{\wedge}{X}_1)\text{-not-}\underset{\wedge}{X}_2 = \underset{\wedge}{X}'\text{-not-}\underset{\wedge}{X}_0$ is a total path to $\underset{\wedge}{x}_j$ in $(\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_1)\text{-not-}\underset{\wedge}{X}_2 = \underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_0$. Conversely, if $\underset{\wedge}{X}''$ is a total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_0 = (\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_1)\text{-not-}\underset{\wedge}{X}_2$, $\underset{\wedge}{X}''$ is by Case $\underline{E}$ the terminal segment of a total path $\underset{\wedge}{X}^*$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_1$ where in turn $\underset{\wedge}{X}^* = \underset{\wedge}{X}'\text{-not-}\underset{\wedge}{X}_1$ by Case $\underline{I}$ for some total path $\underset{\wedge}{X}'$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$. That is, for some total path $\underset{\wedge}{X}'$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$, $\underset{\wedge}{X}''$ is a terminal segment of $\underset{\wedge}{X}'\text{-not-}\underset{\wedge}{X}_1$ and is hence also a terminal segment of $(\underset{\wedge}{X}'\text{-not-}\underset{\wedge}{X}_1)\text{-not-}\underset{\wedge}{X}_2 = \underset{\wedge}{X}'\text{-not-}\underset{\wedge}{X}_0$ since no $\underset{\wedge}{X}_2$-variable is in $\underset{\wedge}{X}''$. In summary, what we have shown is

<u>Theorem</u> 2. Let $\underset{\wedge}{X}_0$ be any tuple of variables in $\underset{\wedge}{X}$, and $\underset{\wedge}{x}_j$ any variable interior to $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_0$ (i.e., $\underset{\wedge}{x}_j$ is any variable in $\underline{I}(\underset{\wedge}{X})$ of which no $\underset{\wedge}{X}_0$-variable is a direct source within $\underset{\wedge}{X}$). Then for each total path $\underset{\wedge}{X}'$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$, some terminal segment $\underset{\wedge}{X}''$ of $\underset{\wedge}{X}'\text{-not-}\underset{\wedge}{X}_0$ is a total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_0$; and each total path $\underset{\wedge}{X}''$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{X}_0$ is a terminal segment of $\underset{\wedge}{X}'\text{-not-}\underset{\wedge}{X}_0$ for some total path $\underset{\wedge}{X}'$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$. If all variables in $\underset{\wedge}{X}_0$ are interior to $\underset{\wedge}{X}$, $\underset{\wedge}{X}''$ is the entirety of the corresponding $\underset{\wedge}{X}'\text{-not-}\underset{\wedge}{X}_0$. <u>Corollary</u>. For any supertuple $\underset{\wedge}{X}^*$ of $\underset{\wedge}{X}$, each path $\underset{\wedge}{X}_{ij}$ from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ is a subtuple of at least one path $\underset{\wedge}{X}^*_{ij}$ from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}^*$, with $\underset{\wedge}{X}^*_{ij}$ containing no $\underset{\wedge}{X}$-variables that are not in $\underset{\wedge}{X}_{ij}$.

<u>Mediational</u> <u>disconnection</u>.

We are now in position to say what it is for one variable to have no effect upon another except through a given tuple of mediators.

<u>Definition</u> 2.8. Variable $\underset{\wedge}{x}_k$ <u>(partially)</u> <u>mediates</u> from variable $\underset{\wedge}{x}_i$ to variable $\underset{\wedge}{x}_j$ iff $\underset{\wedge}{x}_i \neq \underset{\wedge}{x}_k \neq \underset{\wedge}{x}_j$ and $\underset{\wedge}{x}_k$ is on some causal path from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ within some tuple $\underset{\wedge}{Z}$. Tuple $\underset{\wedge}{X}_k$ <u>totally</u> <u>mediates</u> from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ or, equivalently, $\underset{\wedge}{X}_k$ <u>(microstructurally)</u>

<u>disconnects</u> $\mathbf{x}_i$ from $\mathbf{x}_j$ iff (<u>a</u>) $\mathbf{x}_i \neq \mathbf{x}_j$, (<u>b</u>) neither $\mathbf{x}_i$ nor $\mathbf{x}_j$ are in $\mathbf{X}_k$, and (<u>c</u>) for every tuple $\mathbf{Z}$ that includes all of variables $\langle \mathbf{x}_i, \mathbf{X}_k, \mathbf{x}_j \rangle$, every path within $\mathbf{Z}$ from $\mathbf{x}_i$ to $\mathbf{x}_j$ passes through $\mathbf{X}_k$.

This concurs with our initial description of partial mediation (p. 2.5); for ~~there is~~ a path from $\mathbf{x}_i$ through $\mathbf{x}_k$ to $\mathbf{x}_j$ in some $\mathbf{Z}$ just in case $\mathbf{x}_i$ is included in a strictly complete source of $\mathbf{x}_k$ while $\mathbf{x}_k$ in turn is in a strictly complete source of $\mathbf{x}_j$ (see below). And the definition of total mediation is equivalent to saying that when $\mathbf{X}_k$ disconnects $\mathbf{x}_i$ from $\mathbf{x}_j$, $\mathbf{x}_i$ is neither identical with $\mathbf{x}_j$ nor is a direct source of $\mathbf{x}_j$ within any tuple that includes all of $\mathbf{X}_k$. A tighter sense of total mediation could further require $\mathbf{x}_i$ to be a source of each variable in $\mathbf{X}_k$; however, the broader sense given here is technically more advantageous than also requiring $\mathbf{x}_i$ to affect $\mathbf{x}_j$ through $\mathbf{X}_k$'s mediation.

It will later prove to be of great importance that even though total mediation is defined in terms of all paths from $\mathbf{x}_i$ to $\mathbf{x}_j$ in <u>all</u> tuples containing $\langle \mathbf{x}_i, \mathbf{X}_k, \mathbf{x}_j \rangle$, a sufficient condition for $\mathbf{X}_k$ to disconnect $\mathbf{x}_i$ from $\mathbf{x}_j$ can be found in the causal structure within just one of these. Specifically,

<u>Theorem</u> 3. Let $\mathbf{x}_i$, $\mathbf{x}_j$, and variables $\mathbf{X}_k$ be distinct variables in $\mathbf{X}$, with $\mathbf{x}_j$ interior to $\mathbf{X}$. If all paths from $\mathbf{x}_i$ to $\mathbf{x}_j$ in $\mathbf{X}$ pass through $\mathbf{X}_k$, then $\mathbf{X}_k$ disconnects $\mathbf{x}_i$ from $\mathbf{x}_j$ unless $\mathbf{x}_i$ is a source (implicitly--not shown by a path within $\mathbf{X}$) of the first variable in some total path to $\mathbf{x}_j$ within $\mathbf{X}$ that does <u>not</u> pass through $\mathbf{X}_k$. <u>Corollary</u> 1. Tuple $\mathbf{X}_k$ disconnects variable $\mathbf{x}_i$ from variable $\mathbf{x}_j$ whenever $\mathbf{x}_j$ is interior to any tuple that also includes all of $\langle \mathbf{x}_i, \mathbf{X}_k \rangle$ and within which every total path to $\mathbf{x}_j$ as well as every one to $\mathbf{x}_j$ from $\mathbf{x}_i$ passes through $\mathbf{X}_k$. <u>Corollary</u> 2. If $\mathbf{x}_j$ is interior to $\mathbf{X}$, and $\mathbf{X}^*_j$ includes all variables in $\mathbf{x}_j$'s proximal source within $\mathbf{X}$, $\mathbf{X}^*_j$ disconnects each variable in $\mathbf{X}$-not-$\langle \mathbf{X}^*_j, \mathbf{x}_j \rangle$ from $\mathbf{x}_j$.

<u>Proof</u>. Assume the conditions stipulated and hypothesize that within some tuple $\mathbf{Z}$ including all of $\langle \mathbf{x}_i, \mathbf{X}_k, \mathbf{x}_j \rangle$, some path $\mathbf{Z}_{ij}$ from $\mathbf{x}_i$ to $\mathbf{x}_j$ does not pass through $\mathbf{X}_k$. This $\mathbf{Z}$ is then a subtuple of $\langle \mathbf{Z}, \mathbf{X} \rangle$ and this $\mathbf{Z}_{ij}$ is the terminal segment of a

total path to x_j in Z that, by Theorem 2 Corollary, differs from a total path Z' to x_j in $\langle Z,X\rangle$ at most by inclusion in Z' of X-variables not in Z and hence in particular not in X_k. Hence this total path Z' to x_j in $\langle Z,X\rangle$ can be segmented as $Z' = \langle Z_a, Z^*_{ij}\rangle$ wherein terminal segment Z^*_{ij} is a path from x_i to x_j in $\langle Z,X\rangle$ that does not pass through X_k. Now if $Z_0 =_{def}$ Z-not-X comprises just the variables that Z adds to X in $\langle Z,X\rangle$, some terminal segment X'' of Z'-not-Z_0 = $\langle Z_a$-not-Z_0, Z^*_{ij}-not-$Z_0\rangle$ is by Theorem 2 a total path to x_j in X ($\doteq$ $\langle Z,X\rangle$-not-Z_0). X'' cannot include all of Z^*_{ij}-not-Z_0, else Z^*_{ij}-not-Z_0 would be a path from x_i to x_j in X not passing through X_k, contrary to stipulation. So X'' is a terminal segment of Z^*_{ij}-not-Z_0 that is a total path to x_j within X which does not include x_i but begins with some variable x_i' in $\underline{E}(X)$ of which x_i is a source (since some initial segment of Z^*_{ij} is a path from x_i to x_i' in $\langle Z,X\rangle$). So conversely, if x_i is <u>not</u> a source of the first variable in any total path to x_j within X not passing through X_k, there is no Z including all of $\langle x_i, X_k, x_j\rangle$ within which there is some path from x_i to x_j not passing through X_k--i.e., by definition X_k disconnects x_i from x_j. Corollary 1 is immediate; and so is Corollary 2, since in the latter case every path to x_j in X passes through X^*_j. □

Theorem 3 is not a biconditional with the premises given, because even when x_i is a source of some x_i' in $\underline{E}(X)$ from which there is a path to x_j in X not passing through X_k, the $x_i \rightarrow x_i'$ connection too may be wholly mediated by X_k. But it becomes a biconditional if its condition on X is strengthened to say that all total paths to x_j within X pass through X_k.

<u>Mediated regularity</u>: <u>Path principles</u>.

We have assumed without argument that x_i is a source of x_j whenever there is a path from x_i to x_j in some tuple X. But proof is immediate from Theorem 2: If X_0 comprises just the variables between x_i and x_j in some path from x_i to x_j in X, then x_i is a direct source of x_j within X-not-X_0; whereas to the contrary, if there is no path from x_i to x_j in X, x_i is not a direct source of x_j within any subtuple of X. So for any x_i and x_j in X, x_i is included in a strictly complete source of x_j in X just in case there is a path from x_i to x_j in X. (Corollary: x_i is a source of x_j just in case there is a path from x_i to x_j in some tuple X.) We now want to generalize this point to cover complete sources of x_j.

<u>Definition</u> 2.9. Let $\underset{\wedge}{X}_i$ be a subtuple of $\underset{\wedge}{X}$, and $\underset{\wedge}{x}_j$ a variable in $\underline{I}(\underset{\wedge}{X})$-not-$\underset{\wedge}{X}_i$. Then the <u>buffer in</u> $\underset{\wedge}{X}$ (or $\underset{\wedge}{X}$-<u>buffer</u>) from $\underset{\wedge}{X}_i$ to $\underset{\wedge}{x}_j$ is the subtuple $\underset{\wedge}{B}_{ij}$ of $\underset{\wedge}{X}$ comprising just the variables $\{\underset{\wedge}{x}_k\}$ for which some path to $\underset{\wedge}{x}_k$ in $\underset{\wedge}{X}$ passes through $\underset{\wedge}{X}_i$ while some path continuing from $\underset{\wedge}{x}_k$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ does not pass through $\underset{\wedge}{X}_i$.

That is, $\underset{\wedge}{B}_{ij}$ consists of all variables that mediate to $\underset{\wedge}{x}_j$ from the $\underset{\wedge}{X}_i$-variable closest to $\underset{\wedge}{x}_j$ on some path through $\underset{\wedge}{X}_i$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$, combined over all such paths. Evidently, all variables in $\underset{\wedge}{B}_{ij}$ are interior to $\underset{\wedge}{X}$. Hence $\underset{\wedge}{x}_j$ is interior to $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$; and by Theorem 2 Corollary, each total path $\underset{\wedge}{X}''$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$ is $\underset{\wedge}{X}'$-not-$\underset{\wedge}{B}_{ij}$ for some total path $\underset{\wedge}{X}'$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$. This $\underset{\wedge}{X}''$ passes through $\underset{\wedge}{X}_i$ iff $\underset{\wedge}{X}'$ does and moreover, by construction of $\underset{\wedge}{B}_{ij}$, if $\underset{\wedge}{X}''$ passes through $\underset{\wedge}{X}_i$ the direct source of $\underset{\wedge}{x}_i$ within $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$ is the variable in $\underset{\wedge}{X}_i$ closest to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}''$. Consequently, if all total paths to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ pass through $\underset{\wedge}{X}_i$, the variable immediately prior to $\underset{\wedge}{x}_j$ on each total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$ is in $\underset{\wedge}{X}_i$--which is to say that all direct sources of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$ are in $\underset{\wedge}{X}_i$ or, equivalently, that the (non-null) proximal source $\underset{\wedge}{X}^*_j$ of $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$ is a subtuple of $\underset{\wedge}{X}_i$ and hence that $\underset{\wedge}{X}_i$ is an inclusively complete source of $\underset{\wedge}{x}_j$ that is moreover a strictly complete source of $\underset{\wedge}{x}_j$ just in case all $\underset{\wedge}{X}_i$-variables are in $\underset{\wedge}{X}^*_j$. On the other hand, if some total path $\underset{\wedge}{X}''$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$ does not pass through $\underset{\wedge}{x}_j$, $\underset{\wedge}{X}''$ is a subtuple of some total path $\underset{\wedge}{X}'$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ that does not pass through $\underset{\wedge}{X}$ (cf. Theorem 2 Corollary). And if any total path $\underset{\wedge}{X}'$ to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ does not pass through $\underset{\wedge}{X}_i$, then for every subtuple $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ of $\underset{\wedge}{X}$ to which $\underset{\wedge}{x}_j$ is interior, some variable in $\underset{\wedge}{X}'$ and hence <u>not</u> in $\underset{\wedge}{X}_i$ is a direct source of $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$--which is to say that in this case no subtuple of $\underset{\wedge}{X}_i$ is a proximal source of $\underset{\wedge}{x}_j$ in any subtuple of $\underset{\wedge}{X}$ and hence that $\underset{\wedge}{X}_i$ is not an inclusive source of $\underset{\wedge}{x}_j$. To summarize,

<u>Theorem</u> 4. Let $\underset{\wedge}{X}_i$ be any subtuple of $\underset{\wedge}{X}$, $\underset{\wedge}{x}_j$ any variable in $\underline{I}(\underset{\wedge}{X})$ but not in $\underset{\wedge}{X}_i$, and $\underset{\wedge}{B}_{ij}$ the X-buffer from $\underset{\wedge}{X}_i$ to $\underset{\wedge}{x}_j$. Then $\underset{\wedge}{X}_i$ is an inclusively complete source of $\underset{\wedge}{x}_j$ (<u>a</u>) just in case all total paths to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$ pass through $\underset{\wedge}{X}_i$, and also (<u>b</u>) just in case $\underset{\wedge}{X}_i$ includes all variables that are direct sources of $\underset{\wedge}{x}_j$ within $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$. <u>Corollary</u> 1 (from (<u>a</u>)). $\underline{E}(\underset{\wedge}{X})$ is an

inclusively complete source of all variables in $\underline{I}(X)$. <u>Corollary</u> 2 (from ($\underline{b}$)). Under the conditions stipulated, $\underset{\wedge}{X}_i$ is a strictly complete source of $\underset{\wedge}{x}_j$ just in case $\underset{\wedge}{X}_i$ is the proximal source of $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$.

<u>Th.</u>-4 explains how, given just the proximalities in a tuple $\underset{\wedge}{X}$ whose interior includes $\underset{\wedge}{x}_j$, we can proceed to identify whether any given subtuple $\underset{\wedge}{X}_i$ of $\underset{\wedge}{X}$ is an inclusively or strictly complete source of $\underset{\wedge}{x}_j$, namely, by eliminating $\underset{\wedge}{B}_{ij}$ from $\underset{\wedge}{X}$ and observing what proximalities emerge in $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$. This verges upon characterizing how causal regularities that are proximal in $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$ derive from ones that are proximal in $\underset{\wedge}{X}$--except that our postulates so far (<u>CmP</u>-1,2) parse only the qualitative micro-structure of causal mediation without telling how the specific transducers of mediated regularities are determined by the ones from which they derive. To prepare for that story, it helps to re-describe the conversion of proximalities in $\underset{\wedge}{X}$ to proximalities in $\underset{\wedge}{X}$-not-$\underset{\wedge}{B}_{ij}$, or more generally in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ for any subtuple $\underset{\wedge}{X}_0$ of $\underline{I}(\underset{\wedge}{X})$, as a series of intermediate derivations.

Let $\underset{\wedge}{X}_0 = \langle \underset{\wedge}{x}^o_1, \ldots, \underset{\wedge}{x}^o_m \rangle$ be any non-null tuple of variables interior to $\underset{\wedge}{X}$, and write $\underset{\wedge}{X}_1 =_{def} \underset{\wedge}{X}$, $\underset{\wedge}{X}_{k+1} =_{def} \underset{\wedge}{X}_k$-not-$\underset{\wedge}{x}^o_k$ for $\underline{k} = 1, \ldots, \underline{m}$. Then $\underset{\wedge}{X}_1, \ldots, \underset{\wedge}{X}_{m+1}$ is a nested sequence of subtuples of $\underset{\wedge}{X}$ wherein $\underset{\wedge}{X}$ is reduced to $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0 = \underset{\wedge}{X}_{k+1}$ by single-variable deletions and where each variable $\underset{\wedge}{x}_j$ in $\underline{I}(\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_h)$ ($\underline{h} = 1, \ldots, \underline{m}$) has a proximal source $\underset{\wedge}{X}^*_{jk}$ in each $\underset{\wedge}{X}_k$ for which $\underline{k} \leq \underline{h}$. Specifically, $\underset{\wedge}{X}^*_{j(k+1)} \doteq \langle \underset{\wedge}{X}^*_{jk}$-not-$\underset{\wedge}{x}^o_k, \underset{\wedge}{X}^*_{kk} \rangle$ if $\underset{\wedge}{x}_k$ is a direct source of $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}_k$; otherwise, $\underset{\wedge}{X}^*_{j(k+1)} = \underset{\wedge}{X}^*_{jk}$. Different choices of order in $\underset{\wedge}{X}_0$ give different sequences of intermediate proximal sources $\{\underset{\wedge}{X}^*_{jk}\}$; and in particular, if the inversion $\langle \underset{\wedge}{x}^o_m, \underset{\wedge}{x}^o_{m-1}, \ldots, \underset{\wedge}{x}^o_1 \rangle$ of $\underset{\wedge}{X}_0$ is causally well-ordered, each $\underset{\wedge}{x}^o_h$ in $\underset{\wedge}{X}_0$ has the same proximal source in each intermediate stage $\underset{\wedge}{X}_k$ ($\underline{k} \leq \underline{h}$) prior to $\underset{\wedge}{x}^o_h$'s elimination as it has in the original X.

What this stepwise reduction of $\underset{\wedge}{X}$ to $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_0$ shows is simply that when the stage $\underset{\wedge}{X}_k$ is reached for deletion of $\underset{\wedge}{x}^o_k$, the proximal source in $\underset{\wedge}{X}_k$ of each $\underset{\wedge}{x}_j$ becomes $\underset{\wedge}{x}_j$'s proximal source in $\underset{\wedge}{X}_{k+1}$ upon replacing any non-null occurrence of $\underset{\wedge}{x}_k$ therein by $\underset{\wedge}{x}_k$'s own proximal source in $\underset{\wedge}{X}_k$. This is just an application of the composition principle that if $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z}, \underset{\wedge}{z}')$ and $\underset{\wedge}{z}' = \psi(\underset{\wedge}{X})$ are both causal regularities, then there is also a causal regularity $y = \theta(Z,X)$ under which, through the mediation of z', X conjoins Z to determine y. Articulating details of that principle is our next item of business. Meanwhile, in anticipation of CmP-4, below, we can give point to our observations on the sequence of intermediate proximal sources when X is reduced stepwise to X-not-X_0 by letting deletion tuple $X_0 = \langle x_1^o,\ldots,x_m^o\rangle$ be B_{ij} in Theorem 4, and concluding

> Theorem 5. If $x_j = \phi(X_i)$ is a strict causal regularity within X under which subtuple X_i of X is a strictly complete source of x_j, $x_j = \phi(X_i)$ is derivable by iterated composition of mediating causal regularities that are either proximal within X or are themselves derived by composition from ones that are proximal within X. If wanted, the derivation can be a linear sequence $\langle\ldots,\ x_j = \phi_k(X_k),\ x_j = \phi_{k+1}(X_{k+1}),\ldots\rangle$ in which at each step some variable in X_k that mediates between X_i and x_j is replaced by its proximal source in the original tuple X. (Note. Through suitable provisions for augmenting proximal regularities by additional sources of the output that have null weight conjoint with the proximal sources at issue, this composition principle can also be extended to recover all inclusive causal regularities within X from the ones that are proximal within X.)

Mediated regularity: Causal transducers.

What is it to compose one regularity into another? This is virtually the same as composing one function into another except for need to identify not only the resultant regularity's extensional generality but also its transducer (see p. 1.21). For single-argument regularities, the matter is entirely straightforward: The composition of $z = \psi(x)$ into $y = \phi(z)$ is just regularity $y = \phi\psi(x)$ with transducer $\phi\psi$. But more generally, when Z is a tuple of variables that includes z', say $Z = \langle Z_a, z', Z_b\rangle$ where either or both of Z_a and Z_b can be null, the composition of regularity $z' = \psi(X)$ into regularity $y = \phi(Z)$ is the regularity $y = \theta(Z_a, X, Z_b)$ whose transducer θ is defined over all possible values of $W =_{def} \langle Z_a, X, Z_b\rangle$ as follows: Let i_a, i_x, and i_b be subtuples of index sequence $\langle 1,2,\ldots,i,\ldots\rangle$ such that any index i is in

i_a (in i_x, in i_b) iff the $\underline{i}$th variable in W is in Z_a (in X, in Z_b); and for each value $\underline{W}$ of W and index subtuple $\underline{i}$, let $\underline{i}\underline{W}$ be the subtuple of $\underline{W}$ selected by indices i (i.e., the $\underline{i}$th element of $\underline{W}$ is in $\underline{i}\underline{W}$ iff $\underline{i}$ is in i). Then for each value $\underline{W}$ of W, $\theta(\underline{W}) =_{def} \phi(i_a\underline{W}, \psi(i_x\underline{W}), i_b\underline{W})$. To illustrate, suppose that $\phi(Z_a, z', Z_b) = \underline{w}_1 z_a + \underline{w}_2 z' + \underline{w}_3 z_b$ and $\psi(X) = \underline{v}_1 z_a + \underline{v}_2 x$. Then $W = \langle z_a, x, z_b \rangle$, $i_a = \langle 1 \rangle$, $i_x = \langle 1,2 \rangle$, $i_b = \langle 3 \rangle$, and for any value $\underline{W} = \langle \underline{z}_a, \underline{x}, \underline{z}_b \rangle$ of W, $i_a\underline{W} = \langle \underline{z}_a \rangle$, $i_x\underline{W} = \langle \underline{z}_a, \underline{x} \rangle$, $i_b\underline{W} = \langle \underline{z}_b \rangle$; so $\theta(\underline{z}_a, \underline{x}, \underline{z}_b) = \underline{w}_1\underline{z}_a + \underline{w}_2(\underline{v}_1\underline{z}_a + \underline{v}_2\underline{x}) = \underline{w}_3\underline{z}_b = (\underline{w}_1 + \underline{w}_2\underline{v}_1)\underline{z}_a + (\underline{w}_2\underline{v}_2)\underline{x} + \underline{w}_3\underline{z}_b$. This rather tortuous definition of θ is required by cases wherein X has variables in common with $\langle Z_a, Z_b \rangle$, since values of $\langle Z_a, X, Z_b \rangle$ are then not just concatenations of values respectively on Z_a, X, and Z_b. Evidently $\theta(Z_a, X, Z_b) = \phi(Z_a, \psi(X), Z_b)$ when θ is so-defined, while the notation "$\phi(Z_a, \psi(X), Z_b)$" contains within it a full identification of θ in terms of ϕ and ψ. So once the technicalities of transducer composition are clear, we can say simply that the composition of regularity $z' = \psi(X)$ into regularity $y = \phi(Z_a, z', Z_b)$ is regularity $y = \phi(Z_a, \psi(X), Z_b)$.

More generally, whenever we use an expression of form $\underline{\text{Comp}}(\phi_1, \ldots, \phi_m, X_1, \ldots, X_n)$ that defines a composite function on the domain $\underline{P}$ of variables $\langle X_1, \ldots, X_n \rangle$ by simple or recursive compositional combinations of functions $\phi_1, \ldots, \phi_m, X_1, \ldots, X_n$, our notation $\underline{\text{Comp}}(\phi_1, \ldots, \phi_m, X_1, \ldots, X_n)$ also uniquely identifies a function θ from the logical range of $X = \langle X_1, \ldots, X_n \rangle$, i.e. from the set of all possible X-values, onto the range of $\underline{\text{Comp}}(\phi_1, \ldots, \phi_m, X_1, \ldots, X_n)$ such that the value of θ for any argument $\underline{X}$ is the one into which function $\underline{\text{Comp}}(\phi_1, \ldots, \phi_m, X_1, \ldots, X_n)$ <u>would</u> map any member of $\underline{P}$ whose value of X <u>were</u> to be $\underline{X}$. So we can re-conceive $\underline{\text{Comp}}(\phi_1, \ldots, \phi_m, X_1, \ldots, X_n)$ to refer not to the function on $\underline{P}$ that this notation most properly denotes but to the associated transducer θ. Our original composite function on $\underline{P}$ then becomes the composition of X into the re-defined $\underline{\text{Comp}}(\phi_1, \ldots, \phi_m, X_1, \ldots, X_n)$, i.e. into θ; and when we speak of regularity $y = \underline{\text{Comp}}(\phi_1, \ldots, \phi_m, X_1, \ldots, X_n)$, we refer to the 2-tuple comprising first the extensional fact that $y = \theta X$ and secondly the transducer θ.

This explication of regularity compositions also applies to the composition of multiple-output regularity $Z' = \psi(X)$--i.e. $\langle z'_1, \ldots, z'_m \rangle = \langle \psi_1(X), \ldots, \psi_m(X) \rangle$ where

$\underset{\wedge}{Z}' = \langle \underset{\wedge}{z}'_1, \ldots, \underset{\wedge}{z}'_m \rangle$--into regularity $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z})$ when all $\underset{\wedge}{Z}'$-variables are in $\underset{\wedge}{Z}$. But when the $\underset{\wedge}{Z}'$-variables are scattered and reordered in $\underset{\wedge}{Z}$, notation for the general case becomes messy. So for notational simplicity we shall permute as necessary to keep the composition's mediating variables in a compact block. Specifically, if $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z})$ and $\underset{\wedge}{Z}' = \psi(\underset{\wedge}{X})$ with $\underset{\wedge}{Z}'$ essentially identical with a subtuple of $\underset{\wedge}{Z}$, $\underset{\wedge}{Z} = \rho(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{Z}')$ for some permutation operator ρ. Then $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z})$ is logically equivalent to $\underset{\wedge}{y} = \phi\rho(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{Z}')$; and we can stipulate that the composition of $\underset{\wedge}{Z}' = \psi(\underset{\wedge}{X})$ into $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z})$ is $y = \phi\rho(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \psi(\underset{\wedge}{X}))$, the transducer of which is defined by the logic already described. Whenever possible, we shall arrange for ρ to be the Identity permutation.

Having raised the prospect of permuting argument tuples in multiple-input regularities, we had best put on record

> *Causal-mediation Postulate* 3 [*CmP*-3]. If $\underset{\wedge}{Z}$ is a strictly or more generally inclusively complete source of $\underset{\wedge}{y}$ under strict or inclusive causal regularity $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z})$, and tuple $\underset{\wedge}{X}$ is essentially identical with $\underset{\wedge}{Z}$, i.e. $\underset{\wedge}{Z} = \rho(\underset{\wedge}{X})$ for some permutation operator ρ, then $\underset{\wedge}{X}$ is respectively a strictly or inclusively complete source of $\underset{\wedge}{y}$ under causal regularity $\underset{\wedge}{y} = \phi\rho(\underset{\wedge}{X})$.

CmP-3 is not really a substantive postulate, for if $\underset{\wedge}{Z} = \rho(\underset{\wedge}{X})$, "$\underset{\wedge}{y} = \phi(\underset{\wedge}{Z})$" and "$\underset{\wedge}{y} = \phi\rho(\underset{\wedge}{X})$" are essentially just different notations for the same regularity assertion.

Our long-deferred principle of causal composition can now be made explicit as follows:

> *Causal-mediation Postulate* 4 [*CmP*-4]. Let $\underset{\wedge}{x}_j = \phi^*_j(\underset{\wedge}{X}^*_j)$ and $\underset{\wedge}{x}_0 = \phi^*_0(\underset{\wedge}{X}^*_0)$ be proximal regularities within $\underset{\wedge}{X}$, with x_0 one of the variables in $\underset{\wedge}{X}^*_j$, say $\underset{\wedge}{X}^*_j = \langle \underset{\wedge}{X}^*_{aj}, \underset{\wedge}{x}_0, \underset{\wedge}{X}^*_{bj} \rangle$ where either or both of $\underset{\wedge}{X}^*_{aj}$ and $\underset{\wedge}{X}^*_{bj}$ can be null. Then the composition, $y = \phi^*_j(\underset{\wedge}{X}^*_{aj}, \phi^*_0(\underset{\wedge}{X}^*_0), \underset{\wedge}{X}^*_{bj})$ of $\underset{\wedge}{x}_0 = \phi^*_0(\underset{\wedge}{X}^*_0)$ into $\underset{\wedge}{x}_j = \phi^*_j(\underset{\wedge}{X}^*_j)$ is a strict causal regularity under which $\langle \underset{\wedge}{X}^*_{aj}, \underset{\wedge}{X}^*_0, \underset{\wedge}{X}^*_{bj} \rangle$ is the proximal source of $\underset{\wedge}{x}_0$ in any permutation $\rho(\underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0)$ of $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0$ of which $\langle \underset{\wedge}{X}^*_{aj}, \underset{\wedge}{X}^*_0, \underset{\wedge}{X}^*_{bj} \rangle$ is a subtuple.
> *Corollary*. Let $\underset{\wedge}{x}_j$ and $\underset{\wedge}{x}_0$ be interior to $\underset{\wedge}{X} = \langle \underset{\wedge}{X}_a, \underset{\wedge}{x}_0, \underset{\wedge}{X}_b \rangle$ with $\underset{\wedge}{x}_0$ disjoint from

and either $\underset{\wedge}{X}_a$ or $\underset{\wedge}{X}_b$ possibly

$\langle \underset{\wedge}{X}_a, \underset{\wedge}{X}_b \rangle \not\wedge$ null. If $\underset{\wedge}{x}_j = \phi_j(\underset{\wedge}{X})$ and $\underset{\wedge}{x}_0 = \phi_0' \sigma_0(\underset{\wedge}{X})$ are the inclusive causal regularities under which $\underset{\wedge}{X}$ is an inclusively complete source of $\underset{\wedge}{x}_j$ and $\underset{\wedge}{x}_0$, respectively, and in which σ_0 is the subtuple-selector function such that $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0 = \sigma_0(\underset{\wedge}{X})$, then $\langle \underset{\wedge}{X}_a, \underset{\wedge}{X}_b \rangle$ $(= \underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0)$ is an inclusively complete source of $\underset{\wedge}{x}_j$ under inclusive causal regularity $\underset{\wedge}{x}_j = \phi_j(\underset{\wedge}{X}_a, \phi_0'(\underset{\wedge}{X}_a, \underset{\wedge}{X}_b), \underset{\wedge}{X}_b)$.

The "corollary" here is a routine consequence derived by reducing $\underset{\wedge}{x}_j = \phi_j(\underset{\wedge}{X})$ and $\underset{\wedge}{x}_0 = \phi_0' \sigma_0(\underset{\wedge}{X})$ to the strictly causal regularities they embed, composing these by CmP-4, and then re-inserting the remaining variables in $\underset{\wedge}{X}\text{-not-}\underset{\wedge}{x}_0$ with null weights. If $\underset{\wedge}{x}_0$ is not a direct source of $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$, the corollary holds trivially.

CmP-4 seems intuitively obvious, and to avoid lengthening what has already become an unpleasantly turgid story, we shall not here develop the intrinsic argument for it that would be appropriate in a deeper study of causality. We should, however, make clear how CmP-4 differs from simpler but faulty formulations that also seem intuitively to identify mediated causal regularities. And we also need to show that CmP-4 covers all cases wherein identifying which mediated regularities are causal is a problem.

Consider, therefore, the general case of composable strict causal regularities $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z}, \underset{\wedge}{z}_0)$ ($\underset{\wedge}{z}_0$ not in $\underset{\wedge}{Z}$) and $\underset{\wedge}{z}_0 = \psi(\underset{\wedge}{X})$. Evidently $\underset{\wedge}{y}$ and $\underset{\wedge}{z}_0$ are both interior to $\underset{\wedge}{W} =_{def} \langle \underset{\wedge}{y}, \underset{\wedge}{Z}, \underset{\wedge}{z}_0, \underset{\wedge}{X} \rangle$, so $\underset{\wedge}{y}$ and $\underset{\wedge}{z}_0$ both have proximal sources in $\underset{\wedge}{W}$. And since $\underset{\wedge}{z}_0$ is interior to $\underset{\wedge}{W}$, $\underset{\wedge}{y}$ also has a proximal source in $\underset{\wedge}{W}\text{-not-}\underset{\wedge}{z}_0$ which is then also a strictly complete source of $\underset{\wedge}{y}$ in $\underset{\wedge}{W}\text{-not-}\langle \underset{\wedge}{y}, \underset{\wedge}{z}_0 \rangle = \langle \underset{\wedge}{Z}, \underset{\wedge}{X} \rangle$. Accordingly, CmP-4 applies to this general case; and indeed, if $\langle \underset{\wedge}{Z}, \underset{\wedge}{z}_0 \rangle$ and $\underset{\wedge}{X}$ are the respective proximal sources of $\underset{\wedge}{y}$ and $\underset{\wedge}{z}_0$ within $\underset{\wedge}{W}$, CmP-4 says that $\langle \underset{\wedge}{Z}, \underset{\wedge}{X} \rangle$ is a strictly complete source of $\underset{\wedge}{y}$ under (mediated) causal regularity $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z}, \psi(\underset{\wedge}{X}))$. However, the complexities of multivariate causal structure allow that $\langle \underset{\wedge}{Z}, \underset{\wedge}{z}_0 \rangle$ may not be the proximal source of $\underset{\wedge}{y}$, nor $\underset{\wedge}{X}$ of $\underset{\wedge}{z}_0$, in the combined tuple $\underset{\wedge}{W} = \langle \underset{\wedge}{y}, \underset{\wedge}{Z}, \underset{\wedge}{z}_0, \underset{\wedge}{X} \rangle$ even when $\langle \underset{\wedge}{Z}, \underset{\wedge}{z}_0 \rangle$ is a strictly complete source of $\underset{\wedge}{y}$ and $\underset{\wedge}{X}$ of $\underset{\wedge}{z}_0$. And if that is so, while $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z}, \psi(\underset{\wedge}{X}))$ is still a binding of $\underset{\wedge}{y}$ by $\langle \underset{\wedge}{Z}, \underset{\wedge}{X} \rangle$, it does not qualify as a causal regularity under CmP-4--not because CmP-4 is indecisive in this case, but because CmP-4 implies either that the strict causal

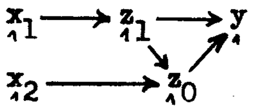


Figure 1.

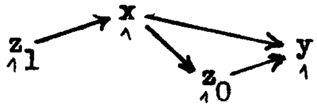


Figure 2.

regularity mapping $\langle \underset{\wedge}{Z}, \underset{\wedge}{X} \rangle$ into $\underset{\wedge}{y}$ has a transducer different from the one in binding $y = \phi(\underset{\wedge}{Z}, \underset{\wedge}{X})$, or, possibly, that only a proper subtuple of $\langle \underset{\wedge}{Z}, \underset{\wedge}{X} \rangle$ is a strictly complete source of $\underset{\wedge}{y}$.

<u>CmP</u>-4 is a carefully restricted special case of a much simpler thesis that on first impression might seem to be all that we need, namely,

> <u>Fallacious</u> <u>Thesis</u> 1 [<u>FT</u>-1]. Let $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z}, \underset{\wedge}{z}_0)$ and $\underset{\wedge}{z}_0 = \psi(\underset{\wedge}{X})$ be strict causal regularities with $\underset{\wedge}{z}_0$ not in $\underset{\wedge}{Z}$. Then $\langle \underset{\wedge}{Z}, \underset{\wedge}{X} \rangle$ is an inclusively (in fact, presumably strictly) complete source of $\underset{\wedge}{y}$ under causal regularity $y = \phi(\underset{\wedge}{Z}, \psi(\underset{\wedge}{X}))$.

<u>FT</u>-1 is so intuitively plausible that I, for one, had long presumed it without suspicion that it might be at all problematic. Yet <u>FT</u>-1 in full generality is incompatible with <u>CmP</u>-1,2,3, as demonstrated by the path structure hypothesized for tuple $\underset{\wedge}{W} = \langle \underset{\wedge}{y}, \underset{\wedge}{z}_1, \underset{\wedge}{z}_0, \underset{\wedge}{x}_1, \underset{\wedge}{x}_2 \rangle$ in Fig. 1. Suppose that the proximal regularities in W for the variables $\langle \underset{\wedge}{y}, \underset{\wedge}{z}_1, \underset{\wedge}{z}_0 \rangle$ comprising $\underset{\wedge}{W}$'s interior are

(2.1) $$\underset{\wedge}{y} = \underline{v}_1 \underset{\wedge}{z}_1 + \underline{v}_0 \underset{\wedge}{z}_0 \,,$$

(2.2) $$\underset{\wedge}{z}_1 = \underline{w}_1 \underset{\wedge}{x}_1 \,,$$

(2.3) $$\underset{\wedge}{z}_0 = \underline{u}_1 \underset{\wedge}{z}_1 + \underline{w}_2 \underset{\wedge}{x}_2 \,,$$

with all coefficients nonzero. Then under <u>CmP</u>-4, the other strict causal regularities in W are

(2.4) $$\underset{\wedge}{y} = (\underline{v}_1 + \underline{v}_0 \underline{u}_1) \underset{\wedge}{z}_1 + (\underline{v}_0 \underline{w}_2) \underset{\wedge}{x}_2 \quad \text{(proximal in } \underset{\wedge}{W}\text{-not-}\underset{\wedge}{z}_0\text{)} \,,$$

(2.5) $$\underset{\wedge}{y} = (\underline{v}_1 \underline{w}_1) \underset{\wedge}{x}_1 + \underline{v}_0 \underset{\wedge}{z}_0 \quad \text{(proximal in } \underset{\wedge}{W}\text{-not-}\underset{\wedge}{z}_1\text{)} \,,$$

(2.6) $$\underset{\wedge}{z}_0 = (\underline{u}_1 \underline{w}_1) \underset{\wedge}{x}_1 + \underline{w}_2 \underset{\wedge}{x}_2 \quad \text{(proximal in } \underset{\wedge}{W}\text{-not-}\underset{\wedge}{z}_1\text{)} \,,$$

(2.7) $$\underset{\wedge}{y} = (\underline{v}_1 + \underline{v}_2 \underline{u}_1) \underline{w}_1 \underset{\wedge}{x}_1 + (\underline{v}_0 \underline{w}_2) \underset{\wedge}{x}_2 \quad \text{(proximal in } \underset{\wedge}{W}\text{-not-}\langle \underset{\wedge}{z}_1, \underset{\wedge}{z}_0 \rangle\text{)} \,.$$

Because the effect of x_1 upon y is mediated entirely by z_1 in Fig. 1, $\langle z_1, x_1, x_2 \rangle$ is not a strictly complete source of y. But $\langle z_1, x_2 \rangle$ is; so by inserting x_1 with null weight into (2.4), we see that

$$(2.8) \qquad y = (\underline{v}_1 + \underline{v}_0\underline{u}_1)z_1 + 0 \cdot x_1 + (\underline{v}_0\underline{w}_2)x_2$$

is the causal regularity under which $\langle z_1, x_1, x_2 \rangle$ is an inclusively complete source of y. On the other hand, it also follows by composition of (2.6) into (2.1) that

$$(2.9) \qquad y = \underline{v}_1 z_1 + (\underline{v}_0\underline{u}_1\underline{w}_1)x_1 + (\underline{v}_0\underline{w}_2)x_2 \ .$$

Regularities (2.9) and (2.8) are just two of many different bindings of y by $\langle z_1, x_1, x_2 \rangle$ that result from the linear dependency in $\langle z_1, x_1, x_2 \rangle$. But (2.9) and (2.8) have different transducers; and since (2.8) is inclusively causal by construction, (2.9) cannot be. Yet under FT-1, (2.9) would qualify as causal because the regularities (2.1) and (2.6) that compose it are strictly causal. This example not merely illustrates the generic untenability of FT-1's claim about causal transducers, but also shows why, when $\langle Z, z_0 \rangle$ and X are strictly complete sources of y and z_0, respectively, the entirety of $\langle Z, X \rangle$ may not be a strictly complete source of y.

FT-1 fails in Fig. 1 because $\langle x_1, x_2 \rangle$ is not the proximal source of z_0 therein. That suggests trying to emend FT-1 as

> Fallacious Thesis 2 [FT-2]. Let $y = \phi(Z, z_0)$ and $z_0 = \psi(X)$ be strict causal regularities with z_0 not in Z. Then if X is the proximal source of z' in $\langle y, Z, z_0, X \rangle$, $\langle Z, X \rangle$ is a strictly complete source of y under causal regularity $y = \phi(Z, \psi(X))$.

But that FT-2, also, is insufficiently constrained is shown by the path structure posited within $W' = \langle y, z_1, z_0, x \rangle$ by Fig. 2. In W', $\langle z_1, z_0 \rangle$ is a strictly complete source of y (albeit not the proximal source of y in W') and x is the proximal source of z_0 in W'; so FT-2 would conclude from composing the determination of z_0 by x into the determination of y by $\langle z_1, z_0 \rangle$ that $\langle z_1, x \rangle$ is a strictly complete source of y. However, intuition and CmP-4 agree to the contrary that $\langle z_1, x \rangle$ is not a strictly complete source of y, insomuch as z_1 affects y in Fig. 2 only through the mediation of x. Even if Fig. 2 were to include a direct-source arrow from z_1 to z_0 so that $\langle z_1, x\rangle$ is indeed a strictly complete source of y, or if <u>FT</u>-2 were weakened to claim only inclusively-complete-causality status for its derived regularity, it is easy to show for linear structural equations that composing $z_2 = \psi(z_1, x)$ into $y = \phi(z_1, z_0)$ assigns the wrong weights (i.e. not the causal ones) to z_1 vs. x in their joint determination of y in this case.

Together, Figs. 1 and 2 illustrate why the full proximality constraints in <u>CmP</u>-4 are needed if composition of one causal regularity into another is to yield a regularity that is also causal.

Demarking which causal compositions are themselves causal becomes even more intricate when, given strict causal regularities $y = \phi(Z, Z')$ and $z'_1 = \psi_1(X_1), \ldots,$ $z'_m = \psi_m(X_m)$ with Z' disjoint from Z and $Z' = \langle z'_1, \ldots, z'_m\rangle$, we wish to find the inclusive, perhaps strict, causal regularity under which y is determined by $\langle Z, X_1, \ldots, X_m\rangle$. <u>CmP</u>-4 does apply to this problem, and what it says to do is this: First, establish the direct-source structure in tuple $W = \langle y, Z, Z', X_1, \ldots, X_m\rangle$ and identify the proximal regularities therein. The latter may or may not include $y = \phi(Z, Z')$ and $\{z'_i = \psi_i(X_i)\}$; if not, the initially given regularities do not suffice to identify the mediated causal regularity we seek. But however we obtain the needed proximals, we then reduce W to W-not-Z' by a sequence $W_{k+1} =_{def} W_k$-not-z''_k ($W_1 =_{def} W$; $\underline{k} = 1, \ldots, \underline{m}$) in which $z''_1, \ldots, z''_m$ is an arbitrary ordering of mediating variables Z'. Every causal regularity that is proximal in W_{k+1} is either also proximal in W_k (cf. <u>CmP</u>-2<u>a</u>) or is identified by <u>CmP</u>-4 from ones that are proximal in W_k; hence the so-identified proximal regularities in $W_k = W$-not-Z' include one whose output is y and whose input is W-not-Z' or a proper subtuple thereof. In fact, if the inversion of $\langle z''_1, \ldots, z''_m\rangle$ is causally well-ordered, i.e. if no z''_i is a source of any z''_j ($\underline{j} > \underline{i}$) later in the composition sequence, every proximal regularity in each W_k ($\underline{k} = 1, \ldots, \underline{m}$) is also proximal in W. Even then it is complicated to write a formula for the derived causal regularity $y = \theta(Z, X_1, \ldots, X_m)$ if some of mediating variables Z' are direct sources of others within W so that some Z'-variables are also in $\langle X_1,\ldots,X_m\rangle$. But if $y = \phi(Z,Z')$ and $z'_i = \psi_i(X_i)$ ($i = 1,\ldots,m$; $Z' = \langle z'_1,\ldots,z'_m\rangle$) are all proximal in W, and Z' is disjoint not only from Z but also from $\langle X_1,\ldots,X_m\rangle$, it is easy to see from CmP-4 by induction on m that $y = \phi(Z,\psi_1(X_1),\ldots,\psi_m(X_m))$ is then a strict causal regularity that is proximal in some permutation of W-not-Z'.

Unhappily, CmP-4's proximality demands are difficult to cope with microstructurally. But CmP-4 does assure us that some compositions of causal regularities preserve causality, and accordingly urges us seek conditions under which this occurs in well-behaved fashion. In general, that search proves feasible only in macrostructural terms and will be pursued later. But one strongly special case is helpful at this point for appraising the practical difference between CmP-4 and FT-1. Suppose that $y = \phi(Z,z')$ (z' not in Z) and $z' = \psi(X)$ are both strictly causal. Then the interior of $W =_{def} \langle y,Z,z',X\rangle$ includes y and z', so E(W) = E(Z,X). Now, $y = \phi(Z,z')$ or $z' = \psi(X)$ fails to be proximal in W only if some x in X-not-$\langle Z,z'\rangle$ is in the W-buffer from $\langle Z,z'\rangle$ to y or some z in Z-not-X is in the W-buffer from X to z'. That requires x or z to be in I(W) and hence cannot occur if variables $\langle Z,X\rangle$ are all in E(W), i.e. if I(Z,X) is null. So

Theorem 6. If $y = \phi(Z,z')$ and $z' = \psi(X)$ are strict causal regularities, their composition $y = \phi(Z,\psi(X))$ is also strictly causal if $\langle Z,X\rangle$ has null interior.

As compositional principles go, Theorem 6 is pretty slim pickings (albeit it has a multiple-mediator generalization--Theorem 22, below--that is rather more impressive). Nevertheless, it prompts the suggestion that so long as we avoid input arrays containing errorless interdependencies, the difference between CmP-4 and FT-1 has little practical significance.

<u>Does</u> <u>FT</u>-1'<u>s</u> <u>defect</u> <u>really</u> <u>matter</u>?

It does indeed. Or at least it <u>should</u>, if our models of multivariate causality have significant application to the real world. Let us accept that we do at times either speculate or estimate empirically that a variable $\underset{\wedge}{y}$ is determined by variables $\langle \underset{\wedge}{Z}, \underset{\wedge}{z}' \rangle$ under some specified causal regularity $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z}, \underset{\wedge}{z}')$, and that by separate hypothesis or experiment we also surmise that $\underset{\wedge}{z}' = \psi(\underset{\wedge}{X})$ is a causal regularity under which input component $\underset{\wedge}{z}'$ in $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z}, \underset{\wedge}{z}')$ is determined by sources of its own. If we have any interest in how $\underset{\wedge}{y}$ is affected by $\underset{\wedge}{X}$, say because we wish to control $\underset{\wedge}{y}$ and can directly manipulate $\underset{\wedge}{X}$ but not $\underset{\wedge}{z}'$, we will almost surely conclude in practice that the force of $\underset{\wedge}{X}$ for $\underset{\wedge}{y}$ conjoint with $\underset{\wedge}{Z}$ is given by the transducer of $\hat{y} = \phi(\hat{Z},\psi(\hat{X}))$. We have seen that this inference is not in principle always correct; but how likely it is to err is another question.

Suspicion that we have little to fear on this score may well be evoked by Theorem 6's suggestion that the problem does not arise so long as we are working with inputs among which there are no errorless dependencies--for prima facie that seems inevitable in practice. Indeed, considering how importantly our theorems in this chapter presuppose not just probabilistic lawfulness but a structure of *complete* causal determinations, one might well wonder if the difference between CmP-4 and FT-1 demarks anything more than the preciosity of an absurdly nonrobust idealization. The present subsection will try to make clear through a simple example that this suspicion is unfounded: So long as we can treat causal-dependency residuals in traditional fashion *as though* they are supplementary sources, violation of Th.-6's exteriority precondition can easily arise in ways more subtle than our usual thinking about these matters is apt to discern.

First, though, let us make the force of what CmP-4 adds to FT-1 more insightful. One point about CmP-4 not yet emphasized adequately is that in order for the composition of strict causal regularities $\hat{y} = \phi(\hat{Z},\hat{z}')$ and $\hat{z}' = \psi(\hat{X})$ to be causal, not only does it suffice under CmP-4 that $y = \phi(\hat{Z},\hat{z}')$ and $\hat{z}' = \psi(\hat{X})$ be proximal in $\hat{W} = \langle\hat{y},\hat{Z},\hat{z}',\hat{X}\rangle$, but this is also virtually necessary. For given that $\langle\hat{Z},\hat{z}'\rangle$ and $\hat{X}$ are complete sources of $\hat{y}$ and $\hat{z}'$, respectively, it follows from CmP-1,2,3 that there are *some* subtuples $\hat{W}_1$ and $\hat{W}_2$ of $\hat{W}$, and transducers ϕ' and ψ', such that $\hat{y} = \phi'(\hat{W}_1,\hat{z}')$ and $\hat{z}' = \psi'(\hat{W}_2)$ are proximal in $\hat{X}$; and only for extraordinarily special parameters in these transducers can resultant causal regularity $\hat{y} = \phi'(\hat{W}_1,\psi'(\hat{W}_2))$ be consistent with $\hat{y} = \phi(\hat{Z},\psi(\hat{X}))$ unless $\hat{W}_1 = \hat{Z}$, $\hat{W}_2 = \hat{X}$, $\phi' = \phi$, and $\psi' = \psi$. As for CmP-4's proximality requirements, observe from Th.-3 Corollary 2 that $\langle\hat{Z},\hat{z}'\rangle$ disconnects each variable in $\hat{X}$-not-$\hat{Z}$ from $\hat{y}$ whenever $\hat{y} = \phi(\hat{Z},\hat{z}')$ is proximal in $\hat{W}$, while conversely, $\hat{y} = \phi(\hat{Z},\hat{z}')$ fails to be proximal in $\hat{W} = \langle\hat{y},\hat{Z},\hat{z}',\hat{X}\rangle$ only if some variable in $\hat{X}$-not-$\hat{Z}$ is a direct source of $\hat{y}$ in $\hat{W}$ and is hence not disconnected from $\hat{y}$ by $\langle\hat{Z},\hat{z}'\rangle$ (cf. Definition 2.8). Similarly, $\hat{z}' = \psi(\hat{X})$ is proximal in $\hat{W}$ just in case

$\underset{\wedge}{X}$ disconnects each variable in $\underset{\wedge}{Z}$-not-$\underset{\wedge}{X}$ from $\underset{\wedge}{z}'$. So CmP-4 can be stated more intuitively, without requiring explicit consideration of proximalities, as

> Theorem 7. If $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z},\underset{\wedge}{z}')$ and $\underset{\wedge}{z}' = \psi(\underset{\wedge}{X})$ are strict causal regularities, their composition $\underset{\wedge}{y} = \phi(\underset{\wedge}{Z},\psi(\underset{\wedge}{X}))$ is also a strict causal regularity if and, virtually, only if $\langle \underset{\wedge}{Z},\underset{\wedge}{z}' \rangle$ disconnects $\underset{\wedge}{y}$ from each $\underset{\wedge}{X}$-variable not in $\underset{\wedge}{Z}$ while $\underset{\wedge}{X}$ disconnects $\underset{\wedge}{z}'$ from each $\underset{\wedge}{Z}$-variable not in $\underset{\wedge}{X}$.

This rewording of the causal-composition principle does not urge the conclusion that violations of its total-mediation precondition are prevalent, but neither does it warrant confidence that violations are rare. As illustrated by Figs. 1 & 2, this all depends on how intricately the variables at issue are causally interwoven. Unless, that is, there is something artifactual about these examples due to their suppression of error terms.

To probe that possibility, envision a structure of causal connections isomorphic to Fig. 2 except for being probabilistic rather than strictly deterministic. Common practice in multivariate causal modeling expresses this by conjecturing the existence of linear structural equations

(2.10) $$\underset{\wedge}{y} = \underline{u}_1\underset{\wedge}{x} + \underline{v}_0\underset{\wedge}{z}_0 + \underset{\wedge}{e}_y \,,$$

(2.11) $$\underset{\wedge}{z}_0 = \underline{w}_1\underset{\wedge}{x} + \underset{\wedge}{e}_0 \,,$$

(2.12) $$\underset{\wedge}{x} = \underline{w}_2\underset{\wedge}{z}_1 + \underset{\wedge}{e}_x \,,$$

in which $\underset{\wedge}{e}_y$, $\underset{\wedge}{e}_0$, and $\underset{\wedge}{e}_x$ are residuals whose nature we leave unspecified except for attributing to them whatever orthogonalities or other distributional properties we need to make the model parameters identifiable. And the conventional digraph representation of structural equations (2.10)-(2.12) is shown in Fig. 3. Presuming that there is an interpretation of these error terms under which the Fig. 3 system behaves <u>as though</u> $\underset{\wedge}{e}_y$, $\underset{\wedge}{e}_0$, and $\underset{\wedge}{e}_x$ are direct sources respectively of $\underset{\wedge}{y}$, $\underset{\wedge}{z}_0$, and $\underset{\wedge}{x}$ in tuple $\underset{\wedge}{W} = \langle \underset{\wedge}{y},\underset{\wedge}{z}_1,\underset{\wedge}{z}_0,\underset{\wedge}{x},\underset{\wedge}{e}_y,\underset{\wedge}{e}_0,\underset{\wedge}{e}_x \rangle$ (the cogency of which presumption we shall examine shortly),

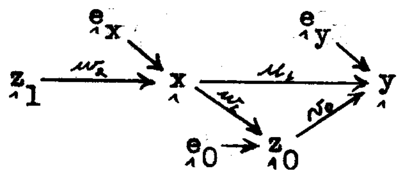


Figure 3.

Fig. 3 then also gives the path structure in $\underset{\wedge}{W}$ as understood in our present sense of this; and by Th.-7 we know that the composition of strict causal regularity (2.12) into strict causal regularity (2.10) is also a strict causal regularity, namely,

(2.13) $$\underset{\wedge}{y} = (\underline{u}_1\underline{w}_2)\underset{\wedge}{z}_1 + \underline{v}_0\underset{\wedge}{z}_0 + [\underline{u}_1\underset{\wedge}{e}_x + \underset{\wedge}{e}_y] \; .$$

Similarly, Th.-7 assures us that

(2.14) $$\underset{\wedge}{y} = (\underline{u}_1 + \underline{v}_0\underline{w}_1)\underset{\wedge}{x} + [\underline{v}_0\underset{\wedge}{e}_0 + \underset{\wedge}{e}_y]$$

and

(2.15) $$\underset{\wedge}{y} = (\underline{u}_1 + \underline{v}_0\underline{w}_1)\underline{w}_2\underset{\wedge}{z}_1 + [(\underline{u}_1 + \underline{v}_0\underline{w}_1)\underset{\wedge}{e}_x + \underline{v}_0\underset{\wedge}{e}_0 + \underset{\wedge}{e}_y]$$

(from (2.11) into (2.10) and (2.12) into (2.14), respectively) are also strict causal regularities.

If variables $\underset{\wedge}{W}' = \langle \underset{\wedge}{y}, \underset{\wedge}{z}_1, \underset{\wedge}{z}_0, \underset{\wedge}{x} \rangle$ are all empirically observable and residuals $\langle \underset{\wedge}{e}_y, \underset{\wedge}{e}_0, \underset{\wedge}{e}_x \rangle$ are all orthogonal to $\underset{\wedge}{W}'$—as we henceforth assume—all coefficients of all data variables in structural equations (2.10)-(2.15) can be identified by ordinary regression analysis (cf. Chapter 3) separately for each equation. Each bracketed compound in equations (2.13)-(2.15) initially appears in the regression solution as a single unanalyzed residual; however, once we have solved for coefficients $\langle \underline{u}_1, \underline{v}_0, \underline{w}_1, \underline{w}_2 \rangle$ and primary residuals $\langle \underset{\wedge}{e}_y, \underset{\wedge}{e}_0, \underset{\wedge}{e}_x \rangle$ in light of the full Fig. 3 structure, we can confirm that the bracketed residuals do decompose as indicated.

On the other hand, the composition of (2.11) into (2.13), namely

(2.16) $$\underset{\wedge}{y} = (\underline{u}_1\underline{w}_2)\underset{\wedge}{z}_1 + (\underline{v}_0\underline{w}_1)\underset{\wedge}{x} + [\underline{v}_0\underset{\wedge}{e}_0 + \underline{u}_1\underset{\wedge}{e}_x + \underset{\wedge}{e}_y]$$

does not qualify as causal under Th.-7; instead, we have from the proximality of

(2.14) in W-not-z_0 that

(2.17) $$y = 0\cdot z_1 + (\underline{u}_1 + \underline{v}_0\underline{w}_1)x + [\underline{v}_0 e_0 + 0\cdot e_x + e_y]$$

is the (inclusive) causal regularity whose transducer maps the input variables in (2.16) into y. Moreover, the coefficients recovered by y's regression upon $\langle z_1, x\rangle$ are the causal weights of these inputs in (2.17) rather than their noncausal ones in (2.16). Yet if we identify just (2.11) and (2.13) by regression, without heed for the larger system, how do we judge that their composition fails to yield causal weights? In particular, why isn't this composition approved under the null-interior precondition of Theorem 6?

Confusion on this point is apt to arise in our treatment of the residuals. When the z_0-mediated composition (2.16) of (2.11) into (2.13) is evaluated for causal status under Th.-6, making clear that (2.16)'s input is the 5-tuple $\langle z_1, x, e_0, e_x, e_y\rangle$ also makes evident, from (2.12), that this input tuple does <u>not</u> have null interior. But if, without regard for all of Fig. 3, we were to identify parameters in (2.11) and (2.13) just by regressing y upon $\langle z_1, z_0\rangle$, and z_0 upon x, we obtain not the entirety of (2.13) but only

(2.17) $$y = (\underline{u}_1\underline{w}_2)z_1 + \underline{v}_0 z_0 + e_c$$

whose residual e_c is a composite

(2.18) $$e_c = \underline{u}_1 e_x + e_y$$

of primary residuals e_x and e_y but is not given to us with that decomposition. Now, the composition of (2.11) into (2.17) is

(2.19) $$y = (\underline{u}_1\underline{w}_2)z_1 + (\underline{v}_0\underline{w}_1)x + \underline{v}_0 e_0 + e_y\ ,$$

the input tuple $\langle z_1, x, e_0, e_c\rangle$ of which does indeed have null interior. So (2.19) would qualify as causal under Th.-6 <u>if</u> its composing regularities (2.11) and (2.17) were both to be causal. But whereas (2.11) is causal by stipulation, we have claimed no general principles under which part of a causal regularity can be treated as a single variable while preserving causal status for the regularity in which it is embedded. What we see here is that (2.13) and (2.17) are indeed not causally equivalent.

The matter cannot be left there, however. For if we could never successfully treat molar abstractions as though they are causal variables in their own right, it is most unlikely that causal models would ever have useful application to the real world. Even in the present example we began by presuming that $\underset{\wedge}{y}$'s partial determination by $\underset{\wedge}{x}$ and $\underset{\wedge}{z}_0$ could be cogently modeled by a strict causal regularity (2.10) in which $\underset{\wedge}{y}$-influences conjoint with but distinct from contributions from $\underset{\wedge}{x}$ and $\underset{\wedge}{z}_0$ are summarized by a single residual $\underset{\wedge}{e}_y$ that behaves for present purposes like a single causal factor. More realistically we should presume only that $\underset{\wedge}{e}_y$ is some logical composite, ideally linear, of an arbitrarily large ensemble $\{\underset{\wedge}{e}_{yi}\}$ of y-sources supplementary to $\langle \underset{\wedge}{x}, \underset{\wedge}{z}_0 \rangle$. But if $\underset{\wedge}{e}_y$ is just shorthand for $\sum_{i=1}^{n} \underline{a}_i e_{yi}$, why is this substitution safe in (2.10) whereas converting (2.13) into (2.17) by substituting $\underset{\wedge}{e}_c$ for $\underline{u}_1 \underset{\wedge}{e}_x + \underset{\wedge}{e}_y$ gets us into FT-1 trouble?

The answer in brief is that if $\underset{\wedge}{e}_y$ (and similarly for $\underset{\wedge}{e}_0$ and $\underset{\wedge}{e}_x$) is replaced by an $\underline{r}$-tuple of supplementary y-sources having the same linkages in the expanded Fig. 3 structure as $\underset{\wedge}{e}_y$ now has, we can replace e_y throughout equations (2.10)-(2.19) by $\sum_{i=1}^{n} \underline{a}_i e_{yi}$ and have everything as before, including in particular which regularities count as strictly or extendedly causal, except that we have no evident way to uncover how many $\underset{\wedge}{e}_{yi}$-variables are composited in $\underset{\wedge}{e}_y$ or what their respective coefficients may be numerically. Alternatively, if we start with

(2.20) $$\underset{\wedge}{y} = \underline{u}_1 \underset{\wedge}{x} + \underline{v}_0 \underset{\wedge}{z}_0 + \sum_{i=1}^{n} \underline{a}_i \underset{\wedge}{e}_{yi}$$

as our postulated structural equation for $\underset{\wedge}{y}$'s determination, and introduce $\underset{\wedge}{e}_y$ as molar abstraction

(2.21) $$\underset{\wedge}{e}_y =_{def} \sum_{i=1}^{n} \underline{a}_i \underset{\wedge}{e}_{yi} ,$$

the structure of mediated causality among the real variables is undisturbed by treating $\underset{\wedge}{e}_y$ as though it is a separate variable, determined (quasi)-causally by the $\underset{\wedge}{e}_{yi}$ under (quasi)-causal regularity (2.20), that totally mediates between $\underset{\wedge}{y}$ and each $\underset{\wedge}{e}_{yi}$-variable. (Precisely <u>why</u> this molar insertion leaves the real structural relations undisturbed in this instance is an important matter that we shall not pursue here.) So long as we are not seeking to identify causal effects on $\underset{\wedge}{y}$ that are mediated by $\underset{\wedge}{e}_y$, we then no more need to include $\underset{\wedge}{e}_y$'s own (quasi)-causal sources in Fig. 3 than we do the sources of $\underset{\wedge}{z}_1$.

But why not treat e_c similarly? There is no objection to that in principle; but the details of this case prevent either of these approaches to the residual in (2.17) from converting (2.17) into a (quasi)-causal regularity from which a causal regularity can be derived by composition with (2.11). If $\underset{\wedge}{e}_c$ is simply replaced by an open tuple $\underset{\wedge}{E}_c = \langle \ldots, \underset{\wedge}{e}_{yi}, \ldots \rangle$ of y-sources supplementary to $\langle \underset{\wedge}{z}_1, \underset{\wedge}{z}_0 \rangle$, we must consider whether $\underset{\wedge}{E}_c$ may not include $\underset{\wedge}{e}_x$ or whatever real supplementary $\underset{\wedge}{x}$-sources are composited in $\underset{\wedge}{e}_x$. Even without special knowledge of the full Fig. 3 structure, we cannot conclude from the lack of linear dependency within $\langle \underset{\wedge}{z}_1, \underset{\wedge}{x}, \underset{\wedge}{e}_c \rangle$ that $\langle \underset{\wedge}{z}_1, \underset{\wedge}{x}, \underset{\wedge}{E}_c$ has null interior. Alternatively, if we treat $\underset{\wedge}{e}_c$ as a molar variable additional to whatever real variables are its quasi-causal sources, it remains to be seen whether any path model for $\langle \underset{\wedge}{W}, \underset{\wedge}{e}_c \rangle$ or some supertuple of $\langle \underset{\wedge}{W}, \underset{\wedge}{e}_c \rangle$ both embeds Fig. 3 and admits (2.17) as (quasi)-causal within $\langle \underset{\wedge}{W}, \underset{\wedge}{e}_c \rangle$.

And in fact none does. There are so many ways to add $\underset{\wedge}{e}_y$ to Fig. 3 that to inventory them here is impractical. But what can be seen is that any path structure envisioned for $\langle \underset{\wedge}{W}, \underset{\wedge}{e}_c \rangle$ either (<u>a</u>) is incompatible under <u>Th</u>.-2 with the Fig. 3 structure for $\underset{\wedge}{W}$ (as occurs e.g. if $\underset{\wedge}{e}_c$ is put on a path from $\underset{\wedge}{e}_x$ to $\underset{\wedge}{y}$ that does not pass through $\underset{\wedge}{x}$ before reaching $\underset{\wedge}{e}_c$), or (<u>b</u>) fails to yield (2.18) even as a binding of $\underset{\wedge}{e}_c$ much less as a (quasi)-causal regularity in $\langle \underset{\wedge}{W}, \underset{\wedge}{e}_c \rangle$ (e.g. if $\underset{\wedge}{e}_c$ is put on a path from $\underset{\wedge}{e}_x$ to $\underset{\wedge}{y}$ that does pass through $\underset{\wedge}{x}$ before reaching $\underset{\wedge}{e}_c$), or (<u>c</u>) achieves (2.17) and (2.18) only as noncausal bindings under constrained model parameters (e.g. when $\underset{\wedge}{e}_y = \underset{\wedge}{e}_c - \underline{u}_1 \underset{\wedge}{e}_x$ is taken to be proximal in $\langle \underset{\wedge}{W}, \underset{\wedge}{e}_c \rangle$ with $\underset{\wedge}{e}_c$ in $\langle \underset{\wedge}{W}, \underset{\wedge}{e}_c \rangle$'s exterior, or when (2.18) is made proximal in $\langle \underset{\wedge}{W}, \underset{\wedge}{e}_c \rangle$ without adding a path from $\underset{\wedge}{e}_c$ to $\underset{\wedge}{y}$). In case (<u>c</u>), composing

(2.11) into (2.17) fails to satisfy the causality precondition of any variant of our causal-composition principle.

The import of this example is threefold. Foremostly, it illustrates why explicit acknowledgement of causal residuals does not undermine the account of causal structure here developed in terms of errorless regularities. In particular, it explains why interiority is more likely to jeopardize causal interpretation of bindings derived by composition from other prima facie causal regularities than is evident from just the joint distribution of data variables and regression residuals. But beyond that, the example urges appreciation of how tricky it can be to interpret residuals causally, and further demonstrates that we cannot arbitrarily treat molar composites as though they are causal factors in their own right without disrupting the causal story we are trying to put together. In later chapters here we shall have more to say about the practicalities of analyzing residuals. But how best to treat molar abstractions as conceptually distinct factors interwoven with real variables in a coherent quasi-causal generalization of molecular causality is a foundational theory whose pervasive neglect we cannot aspire to redress on this occasion.

<u>Null weights vs. zero weights</u>.

When introducing the concept of proximality, we distinguished between strict causal regularities and inclusive ones that are not strict in terms of the latter containing input variables that are given "null" weight by the regularity's transducer. Specifically, $\underset{\wedge}{y} = \phi(\underset{\wedge}{X})$ is an inclusive but not strict causal regularity just in case (<u>a</u>) a proper subtuple $\underset{\wedge}{X}^*$ of $\underset{\wedge}{X}$ is a strictly complete source of $\underset{\wedge}{y}$ under some causal regularity $\underset{\wedge}{y} = \phi^*(\underset{\wedge}{X}^*)$ and (<u>b</u>) $\phi = \phi^*\sigma$ for the subtuple-selector function σ that picks $\underset{\wedge}{X}^*$ out of $\underset{\wedge}{X}$. For reasons explained earlier (p. 2.15), we can then say that the variables <u>not</u> in $\underset{\wedge}{X}$'s subtuple $\sigma(\underset{\wedge}{X})$ have null weight in $\underset{\wedge}{y} = \phi^*\sigma(\underset{\wedge}{X})$. It would be highly convenient to assert that conversely, whenever $\underset{\wedge}{y} = \phi(\underset{\wedge}{X})$ is a <u>strict</u> (i.e. nomically irreducible) causal regularity, there is no subtuple-selector σ

for which $\sigma(X)$ omits part of X while $\phi = \phi^*\sigma$ for some transducer ϕ^*. That would be true if nomically irreducible causal regularities were always functionally irreducible as well (cf. p. 1.9). But unhappily for simplicity, that is not the case--at least not in principle.

Consider again the path structure in Fig. 3 for structural equations (2.10)-(2.12) and their compositional consequences. Since by stipulation (2.10) and (2.11) are causal regularities that are not just strict but proximal in Fig. 3, principle CmP-4 entails that (2.14) too is a strict causal regularity. Now, there is nothing in this model's open parameters to preclude the numerical value of path coefficient u_1 happening to equal the negated product of path coefficients v_0 and w_1. Yet if u_1 does equal $-v_0 w_1$, ~~does equal~~ (2.14) becomes

(2.22) $$y = 0 \cdot x + v_0 e_0 + e_y \qquad (u_1 = -v_0 w_0)\ .$$

This is not the same as

(2.23) $$y = v_0 e_0 + e_y\ ;$$

for not only do (2.22) and (2.23) have different transducers--one is a function on the logical range of $\langle x, e_0, e_y \rangle$, the other only on that of $\langle e_0, e_y \rangle$--but also (2.22) qualifies as strictly causal under CmP-4 even when x's coefficient turns out to be numerically zero whereas (2.23) is a happenstance binding of y by $\langle e_0, e_y \rangle$ that cannot be counted as causal without disrupting the strict-causality character of (2.22). It follows that we must distinguish in inclusive causal regularities between null weights and numerically zero weights that are not null. A variable x_i having null weight in $y = \phi(X)$ makes no causal contribution to y except through the mediation of variables X-not-x_i. But if x_i's weight in $y = \phi(X)$ is a non-null zero, x_i does have an independent effect on y conjoint with X-not-x_i even if only one that is negligible to the highest degree.

In light of possibilities like (2.22), it would be preferable to define the concept of causal transducer in a way that distinguishes null weights from zero weights in causal regularities that are functionally reducible. But that opens the broader question whether the modern set-theoretic construal of functions does sufficient justice to the ontological character of transducers in natural regularities--an issue that we can best shun on this occasion. Meanwhile, if the prospect of causal weights that are zero but not null occasions distress, it will surely do little harm to posit that as a matter of brute fact, no extended causal regularity $\underset{\wedge}{y} = \phi(\underset{\wedge}{X})$ in our real world happens to give exactly zero weight to any variable in the subtuple of $\underset{\wedge}{X}$ that is a strictly complete source of $\underset{\wedge}{y}$. Who can show otherwise?

Causal Macrostructure.

In molar models of causality, we conceive of molar variables $\{\tilde{x}_i\}$ that are logical abstractions $\underset{\wedge}{\tilde{x}_i} =_{def} \chi_i(\underset{\wedge}{X_i})$ (not always recognized as such) from underlying ensembles $\{\underset{\wedge}{X_i}\}$ of molecular variables, and seek to find regularities governing the $\underset{\wedge}{\tilde{x}_i}$ that are isomorphic or at least homomorphic to causal determinations among the tuples $\underset{\wedge}{X_i}$ they respectively reflect. A distinguished special case of molar causality that is both propaedeutic for the general theory and of value to multivariate modeling in its own right arises when the molar units are themselves tuples of the variables whose causal microstructure is to be abstracted. Somewhat arbitrarily, we shall adopt the label "causal macrostructure" for this case and define it as the theory of causal relations among Tuples, where "Tuple" is henceforth shorthand for "tuples of variables" in the special sense stipulated at this Chapter's outset." (Whenever context permits, however, we shall demote the capital in "Tuple" to lowercase.)

Basically, the theory of causal macrostructure seeks to identify partial-order relations among Tuples that usefully capture our intuitive appraisals of one multivariate complex being causally antecedent to another, and which unfold into models of multivariate mediation that subsumes microcausal path structure as a limiting case while allowing us to think more generally about causal relations among groups of variables in the same formal terms that are effective for simple cases of microstructural causality. At the core of any such theory must lie multivariate generalizations of causal-source relations on single variables. This means that ideally, i.e. perhaps with certain qualifications that do not significantly degrade the microstructural parallel, we want to define binary relations $\Rightarrow$ and $\rightarrow$ on Tuples such that: (*a*) $\underset{\wedge}{X} \Rightarrow \underset{\wedge}{Y}$ just in case tuple $\underset{\wedge}{X}$ causally or quasi-causally determines tuple $\underset{\wedge}{Y}$ in a conceptually natural extension of strictly complete microcausal regularity. (*b*) $\underset{\wedge}{X} \rightarrow \underset{\wedge}{Y}$ just in case $\langle \underset{\wedge}{X}, \underset{\wedge}{Z} \rangle \Rightarrow \underset{\wedge}{Y}$ for some possibly-null supplementary tuple $\underset{\wedge}{Z}$ (so that $\underset{\wedge}{X} \Rightarrow \underset{\wedge}{Y}$ implies $\underset{\wedge}{X} \rightarrow \underset{\wedge}{Y}$ though not conversely) and reduces to the causal-source relation between single variables when $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$ are singleton tuples. And (*c*) our macrostructural $\rightarrow$ is to have essentially the same partial-order properties over its full domain of Tuples as it does when restricted just to singletons --which entails that $\Rightarrow$, too, must be a partial order on Tuples. We also want our multivariate version of the strictly-complete-source relation to have the qualitative compositional property (*d*) that if $\langle \underset{\wedge}{Z}, \underset{\wedge}{Z}' \rangle \Rightarrow \underset{\wedge}{Y}$ and $\underset{\wedge}{X} \Rightarrow \underset{\wedge}{Z}'$ then $\langle \underset{\wedge}{Z}, \underset{\wedge}{X} \rangle \Rightarrow \underset{\wedge}{Y}$. (Roughly speaking, this is the macrocausal counterpart of Th.-1*a* (p. 2.10), which is the heart of microcausal path structure.

Much of the work for any account of causal macrostructure is ascertaining which relations defined over Tuples in terms of causal connection among their constituents have the partial-order character of causality. So we had best begin by formalizing the order properties at issue, especially since the essential identity ($\doteq$) of Tuples differing only by permutation requires us to use a sense of partial order slightly more complicated than the standard definition of this.

Definition 2.10. Let *R* be a binary relation on Tuples. Then *R* is *transitive* iff $\underset{\wedge}{X}R\underset{\wedge}{Z}$ whenever $\underset{\wedge}{X}R\underset{\wedge}{Y}$ and $\underset{\wedge}{Y}R\underset{\wedge}{Z}$, *reflexive* iff always $\underset{\wedge}{X}R\underset{\wedge}{X}$, *irreflexive* iff $\underset{\wedge}{X}R\underset{\wedge}{X}$ only when $\underset{\wedge}{X}$ is null, *symmetric* iff $\underset{\wedge}{Y}R\underset{\wedge}{X}$ whenever $\underset{\wedge}{X}R\underset{\wedge}{Y}$, *anti-symmetric* relative to some equivalence relation $\cong$ iff both $\underset{\wedge}{X}R\underset{\wedge}{Y}$ and $\underset{\wedge}{Y}R\underset{\wedge}{X}$ only when $\underset{\wedge}{X} \cong \underset{\wedge}{Y}$, and *classically anti-symmetric* iff it is anti-symmetric relative just to $\doteq$. Relation *R* is a *partial order* relative to equivalence relation $\cong$ iff it is transitive and anti-symmetric relative to $\cong$, a *classical partial order* iff it is a partial order relative just to $\doteq$, and a *strict partial order* iff it is both transitive and irreflexive.

If *R* is a strict partial order, i.e. is transitive and irreflexive, then *R* is anti-symmetric relative to every ≅ and is hence also a partial order relative to every ≅. For if ever both $X\underline{R}Y$ and $Y\underline{R}X$ for any such $\underline{R}$, it follows by transitivity that $X\underline{R}X$ and $Y\underline{R}Y$, hence X and Y are null by irreflexivity and so X ≅ Y for any ≅. (After the model of null sets, we stipulate that there is only one null tuple; hence X = Y whenever X and Y are null from the definitional reflexivity of equivalence relations.)

Many partial-order relations on Tuples can be defined from causal connections among their constituents, albeit not all are equally useful. A basic pair is

<u>Definition</u> 2.11. A tuple X of variables <u>b(roadly)</u> <u>precedes</u> tuple Y iff each variable in Y has a source in X. Tuple X <u>t(ightly)</u> <u>precedes</u> tuple Y iff each variable in X is a source of some variable in Y. (Note the duality of broad and tight precedence.)

For singleton Tuples, broad and tight precedence both reduce to the causal-source relation. Specifically, x is a source of y iff ⟨x⟩ b-precedes ⟨y⟩ and also iff ⟨x⟩ t-precedes ⟨y⟩. Although X can precede Y both broadly and tightly even when some variables in Y are sources of variables in X, the broad and tight precedence relations are nevertheless both strict partial orders.

<u>Proof</u>. If X b-precedes Y and Y b-precedes Z, each z in Z has as source some y in Y that in turn has a source in X; so by the transitivity of the causal-source relation, each z in Z has a source in X--i.e., b-precedence is transitive. And if any tuple X were to b-precede itself, we could construct an arbitrarily long sequence of variables in X, each of which is a source of all variables that follow it in the sequence. Since X is finite, some variable would eventually have to recur in this sequence, violating the causal-source relation's irreflexivity. So b-precedence must also be irreflexive. The transitivity and irreflexivity of t-precedence follows similarly (by duality).

When X broadly precedes Y, each variable in Y is causally influenced by some part of $\underset{\wedge}{X}$. If those influences are all complete determinations, we have the paradigm of errorless multiple-output causality. However, to catch the multivariate causal ordering that results from replacing just part of a tuple of variables by sources of that part, we want a sense of quasi-causal determination under which, if $\underset{\wedge}{X}$ determines $\underset{\wedge}{Y}$, then $\langle \underset{\wedge}{X},\underset{\wedge}{Z}\rangle$ determines $\langle \underset{\wedge}{Y},\underset{\wedge}{Z}\rangle$ for any additional tuple $\underset{\wedge}{Z}$ regardless of how $\underset{\wedge}{Z}$ may or may not be related to $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$. Much of our need in that respect is nicely served by

> <u>Definition</u> 2.12. A tuple $\underset{\wedge}{X}$ of variables <u>s(tructurally) determines</u> tuple $\underset{\wedge}{Y}$--symbolized $\underset{\wedge}{X} \dot{\Rightarrow} \underset{\wedge}{Y}$--iff, for each variable $\underset{\wedge}{y}_i$ in $\underset{\wedge}{Y}$-not-$\underset{\wedge}{X}$, some subtuple $\underset{\wedge}{X}_i$ of $\underset{\wedge}{X}$ is a strictly complete source of $\underset{\wedge}{y}_i$. (This is true vacuously if $\underset{\wedge}{Y}$-not-$\underset{\wedge}{X}$ is null, i.e. if all $\underset{\wedge}{Y}$-variables are in $\underset{\wedge}{X}$.) Tuples $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$ are <u>s(tructurally) interderivable</u> ($\underset{\wedge}{X} \dot{\Leftrightarrow} \underset{\wedge}{Y}$) iff $\underset{\wedge}{X}$ s-determines $\underset{\wedge}{Y}$ and $\underset{\wedge}{Y}$ s-determines $\underset{\wedge}{X}$.

It will be evident that if $\underset{\wedge}{X} \dot{\Rightarrow} \underset{\wedge}{Y}$, then (<u>a</u>) $\langle \underset{\wedge}{X},\underset{\wedge}{Z}\rangle \dot{\Rightarrow} \langle \underset{\wedge}{Y},\underset{\wedge}{Z}\rangle$ for any $\underset{\wedge}{Z}$, (<u>b</u>) $\underset{\wedge}{X} \dot{\Rightarrow} \underset{\wedge}{Y}'$ for any subtuple $\underset{\wedge}{Y}'$ of $\underset{\wedge}{Y}$, and (<u>c</u>) $\underset{\wedge}{X} \dot{\Rightarrow} \underset{\wedge}{Y}'$ if $\underset{\wedge}{Y}' \doteq \underset{\wedge}{Y}$. For an example of s-interderivability, suppose that $\underset{\wedge}{x}$ is a strictly complete source of $\underset{\wedge}{y}$ which in turn is a strictly complete source of $\underset{\wedge}{z}$. Then $\underset{\wedge}{x}$ is also a strictly complete source of $\underset{\wedge}{z}$, so $\langle \underset{\wedge}{x},\underset{\wedge}{y}\rangle \dot{\Leftrightarrow} \langle \underset{\wedge}{x},\underset{\wedge}{z}\rangle$ even though there is an intuitive causal-order asymmetry between these two 2-tuples.

It is useful to observe that

> <u>Theorem</u> 8. Tuple $\underset{\wedge}{X}$ s-determines tuple $\underset{\wedge}{Y}$ just in case $\underline{E}(\underset{\wedge}{X}) \doteq \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y})$ (equivalently, just in case $\underline{E}(\underset{\wedge}{X}) = \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y})$). <u>Corollary</u>. If $\underline{E}(\underset{\wedge}{X}) \doteq \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y})$, then (<u>a</u>) $\underline{E}(\underset{\wedge}{X},\underset{\wedge}{Z}) \doteq \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y},\underset{\wedge}{Z})$ for any tuple $\underset{\wedge}{Z}$, and (<u>b</u>) $\underline{E}(\underset{\wedge}{X}) \doteq \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y}')$ for any subtuple $\underset{\wedge}{Y}'$ of $\underset{\wedge}{Y}$.

<u>Proof</u>. We are to show that $\underset{\wedge}{X} \dot{\Rightarrow} \underset{\wedge}{Y}$ if and only if $\langle \underset{\wedge}{X},\underset{\wedge}{Y}\rangle$ and $\underset{\wedge}{X}$ have the same exterior. Let $\underset{\wedge}{Z} =_{def} \langle \underset{\wedge}{X},\underset{\wedge}{Y}\rangle$ while $\underset{\wedge}{Z}_0 =_{def} \underset{\wedge}{Y}$-not-$\underset{\wedge}{X}$. If $\underset{\wedge}{X} \dot{\Rightarrow} \underset{\wedge}{Y}$, each $\underset{\wedge}{Z}$-variable has a strictly complete source in $\underset{\wedge}{X}$ and hence in $\underset{\wedge}{Z}$, so all $\underset{\wedge}{Z}_0$-variables are interior to $\underset{\wedge}{Z}$. Then by <u>Th.-1a</u> Corollary, $\underline{E}(\underset{\wedge}{Z}$-not-$\underset{\wedge}{Z}_0) = \underline{E}(\underset{\wedge}{Z})$, or $\underline{E}(\underset{\wedge}{X}) = \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y})$ since $\underset{\wedge}{Z}$-not-$\underset{\wedge}{Z}_0 = \underset{\wedge}{X}$. Conversely, in order to have $\underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y}) \doteq E(\underset{\wedge}{X})$, i.e. $\underline{E}(\underset{\wedge}{Z}) \doteq \underline{E}(\underset{\wedge}{Z}$-not-$\underset{\wedge}{Z}_0)$, all $\underset{\wedge}{Z}_0$-variables must be in $\underline{I}(\underset{\wedge}{Z})$ (since otherwise some $\underset{\wedge}{z}_0$ in $\underline{E}(\underset{\wedge}{Z})$ would not be in $\underset{\wedge}{Z}$-not-$\underset{\wedge}{Z}_0$ and hence

not in $\underline{E}(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z_0})$), whence by Th.-4 Corollary 1, each variable in $\underset{\wedge}{Z_0}$, i.e. $\underset{\wedge}{Y}\text{-not-}\underset{\wedge}{X}$, has a strictly complete source in $\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z_0}$, i.e. $\underset{\wedge}{X}$, so that $\underset{\wedge}{X}$ s-determines $\underset{\wedge}{Y}$. The corollary follows from observations (a,b) immediately following Def. 2.12.

Now suppose that $\underset{\wedge}{X} \overset{\cdot}{\Rightarrow} \underset{\wedge}{Y}$ and $\underset{\wedge}{Y} \overset{\cdot}{\Rightarrow} \underset{\wedge}{Z}$. Then $\underline{E}(\underset{\wedge}{X}) = \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y})$ and $\underline{E}(\underset{\wedge}{Y}) = \underline{E}(\underset{\wedge}{Y},\underset{\wedge}{Z})$ while from the latter $\underline{E}(\underset{\wedge}{Y},\underset{\wedge}{X}) = \underline{E}(\underset{\wedge}{Y},\underset{\wedge}{Z},\underset{\wedge}{X})$; so $\underline{E}(\underset{\wedge}{X}) \doteq \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y},\underset{\wedge}{Z})$ or $\underset{\wedge}{X} \overset{\cdot}{\Rightarrow} \langle \underset{\wedge}{Y},\underset{\wedge}{Z} \rangle$ and hence $\underset{\wedge}{X} \overset{\cdot}{\Rightarrow} \underset{\wedge}{Z}$. This proves

<u>Theorem</u> 9. If $\underset{\wedge}{X}$ s-determines $\underset{\wedge}{Y}$ and $\underset{\wedge}{Y}$ s-determines $\underset{\wedge}{Z}$, then $\underset{\wedge}{X}$ s-determines $\underset{\wedge}{Z}$.

That is, s-determination is transitive. Beyond that, however, it is a partial order only relative to s-interderivability. Although that is no problem for many purposes, causal-order distinctions within $\overset{\cdot}{\Leftrightarrow}$-equivalence classes also need recognition. We have already noted one example of s-interderivable Tuples that are causally asymmetric. Another instance: If $\underset{\wedge}{X}$ is a strictly complete source of each variable in $\underset{\wedge}{Z}$ while $\underset{\wedge}{X}$ and $\underset{\wedge}{Z}$ together are a strictly complete source of $\underset{\wedge}{y}$, to acknowledge macrostructurally that $\langle \underset{\wedge}{X},\underset{\wedge}{Z} \rangle$ mediates between $\underset{\wedge}{X}$ and $\underset{\wedge}{y}$ we must identify the sense in which $\underset{\wedge}{X}$ is causally prior to $\langle \underset{\wedge}{X},\underset{\wedge}{Z} \rangle$ even though $\underset{\wedge}{X} \overset{\cdot}{\Leftrightarrow} \langle \underset{\wedge}{X},\underset{\wedge}{Z} \rangle$.

The microstructural nature of s-interderivability is plain enough from Theorem 8: If $\underset{\wedge}{X} \overset{\cdot}{\Rightarrow} \underset{\wedge}{Y}$ and $\underset{\wedge}{Y} \overset{\cdot}{\Rightarrow} \underset{\wedge}{X}$, then $\underline{E}(\underset{\wedge}{X}) \doteq \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y}) \doteq \underline{E}(\underset{\wedge}{Y},\underset{\wedge}{X}) \doteq \underline{E}(\underset{\wedge}{Y})$. That is, any $\overset{\cdot}{\Leftrightarrow}$-equivalence class consists of Tuples whose exteriors are essentially identical to one another. So if $\underset{\wedge}{X} \overset{\cdot}{\Leftrightarrow} \underset{\wedge}{Y}$, any finer-grained ordering of $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$ must reflect some causal asymmetry between $\underline{I}(\underset{\wedge}{X})$ and $\underline{I}(\underset{\wedge}{Y})$. One possibility might be to say that $\underset{\wedge}{X}$ is prior to $\underset{\wedge}{Y}$ if $\underset{\wedge}{X} \overset{\cdot}{\Rightarrow} \underset{\wedge}{Y}$ but not $\underset{\wedge}{Y} \overset{\cdot}{\Rightarrow} \underset{\wedge}{X}$, or if $\underline{I}(\underset{\wedge}{X}) \overset{\cdot}{\Rightarrow} \underline{I}(\underset{\wedge}{Y})$ but not conversely when $\underset{\wedge}{X} \overset{\cdot}{\Leftrightarrow} \underset{\wedge}{Y}$, or if $\underline{I}(\underline{I}(\underset{\wedge}{X})) \overset{\cdot}{\Rightarrow} \underline{I}(\underline{I}(\underset{\wedge}{Y}))$ but not conversely when $\underset{\wedge}{X} \overset{\cdot}{\Leftrightarrow} \underset{\wedge}{Y}$ and $\underline{I}(\underset{\wedge}{X}) \overset{\cdot}{\Leftrightarrow} \underline{I}(\underset{\wedge}{Y})$, etc. That handles our first example of s-interderivability (i.e., between $\langle \underset{\wedge}{x},\underset{\wedge}{y} \rangle$ and $\langle \underset{\wedge}{x},\underset{\wedge}{z} \rangle$ when $\underset{\wedge}{x}$ is a strictly complete source of $\underset{\wedge}{y}$ and $\underset{\wedge}{y}$ one of $\underset{\wedge}{z}$). But it fails to make $\underset{\wedge}{x}$ prior to $\langle \underset{\wedge}{x},\underset{\wedge}{y} \rangle$ in our second test case where $\underset{\wedge}{x}$ is a strictly complete source of $\underset{\wedge}{y}$.

The basic reason why s-determination misses the intuitive asymmetry between (some) $\overset{\cdot}{\Leftrightarrow}$-equivalent Tuples as in our two examples is that it is in effect an expanded version of <u>broad</u> precedence, i.e. it allows the antecedent of $\underset{\wedge}{X} \stackrel{\bullet}{\Rightarrow} \underset{\wedge}{Y}$ to contain variables that are irrelevant to its consequent. But whereas all variables in $\underset{\wedge}{X}$ are logically or causally relevant to $\langle \underset{\wedge}{X},\underset{\wedge}{Y} \rangle$ when $\underset{\wedge}{X}$ is a strictly complete source of (all of) $\underset{\wedge}{Y}$, the $\underset{\wedge}{Y}$-part of $\langle \underset{\wedge}{X},\underset{\wedge}{Y} \rangle$ is not correspondingly relevant to $\underset{\wedge}{X}$. More generally, if $\underset{\wedge}{X}$ is a complete source of both $\underset{\wedge}{Y}$ and $\underset{\wedge}{Z}$, so that $\langle \underset{\wedge}{X},\underset{\wedge}{Y} \rangle$ and $\langle \underset{\wedge}{X},\underset{\wedge}{Z} \rangle$ are both s-interderivable, $\langle \underset{\wedge}{X},\underset{\wedge}{Y} \rangle$ is intuitively prior to $\langle \underset{\wedge}{X},\underset{\wedge}{Z} \rangle$ if all of $\underset{\wedge}{X}$ is relevant to $\underset{\wedge}{Z}$ with all of $\underset{\wedge}{Y}$ mediating between $\underset{\wedge}{X}$ and $\underset{\wedge}{Z}$, but not if $\underset{\wedge}{Y}$ and $\underset{\wedge}{Z}$ are independent effects of $\underset{\wedge}{X}$. To formalize this intuition, we need a relaxation of the tight-precedence relation that leaves unconstrained the variables its relata are allowed to have in common. Specifically, let us say

<u>Definition</u> 2.13. A tuple $\underset{\wedge}{X}$ is <u>t(ightly) prior</u> to tuple $\underset{\wedge}{Y}$ iff there is a possibly-null tuple $\underset{\wedge}{Z}$ containing just variables common to $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$ such that $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$ tightly precedes $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$. (By duality, $\underset{\wedge}{X}$ is <u>b(roadly) prior</u> to $\underset{\wedge}{Y}$ iff, for a tuple $\underset{\wedge}{Z}$ of variables common to $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$, $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$ broadly precedes $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$. However, we shall have no interest in b-priority.)

This definition is equivalent to what is prima facie a much stronger condition, namely,

<u>Theorem</u> 10. Tuple $\underset{\wedge}{X}$ is t-prior to tuple $\underset{\wedge}{Y}$ just in case $\underset{\wedge}{X}$-not-$\underset{\wedge}{Y}$ t-precedes $\underset{\wedge}{Y}$-not-$\underset{\wedge}{X}$.

<u>Proof</u>. That the right-hand side of this biconditional entails its left-hand side is evident from the definition of t-priority. For the converse, let $\underset{\wedge}{Z}^*$ consist of all the variables common to $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$, and let $\underset{\wedge}{Z}$ be any subtuple of $\underset{\wedge}{Z}^*$ such that $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$ t-precedes $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$. If $\underset{\wedge}{Z}^* = \underset{\wedge}{Z}$, the converse is immediate. Otherwise, let $\underset{\wedge}{Z}' = \langle \underset{\wedge}{z}'_1,\ldots,\underset{\wedge}{z}'_m \rangle$ ($m \geq 1$) be $\underset{\wedge}{Z}^*$-not-$\underset{\wedge}{Z}$ permuted to be causally well-ordered. Then $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$ comprises just the variables in $\underset{\wedge}{Y}$-not-$\underset{\wedge}{X}$ together with those in $\underset{\wedge}{Z}'$, while $\underset{\wedge}{X}$-not-$\underset{\wedge}{Y}$ is subtuple ($\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$)-not-$\underset{\wedge}{Z}'$ of $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$. By assumption that $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$ t-precedes $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$, we have that $\underset{\wedge}{z}'_{m-i}$ t-precedes (i.e. is a source of some variable in) $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$ $\doteq \langle \underset{\wedge}{Y}\text{-not-}\underset{\wedge}{X},\underset{\wedge}{Z}' \rangle$ for each $i = 0,1,\ldots,m-1$. For $i = 0$, by the causal well-ordering of

Z', z'_m is not a source of any variable in Z' and so must t-precede Y-not-X. More generally by the well-ordering, z'_{m-i} must t-precede ⟨$z'_{m-i+1}, \ldots, z'_m$, Y-not-X⟩ and hence (by transitivity of the causal-source relation) t-precedes Y-not-X if each variable after z'_{m-i} in Z' t-precedes Y-not-X. So by induction on i, Z' t-precedes Y-not-X-- which by the transitivity of t-precedence evidently entails that any Tuple which t-precedes Y-not-Z ≐ ⟨Y-not-X, Z'⟩ also t-precedes Y-not-X. So given that X-not-Z and hence its subtuple X-not-Y t-precedes Y-not-Z, it follows that X-not-Y t-precedes Y-not-X. □

Theorem 11 says that X is t-prior to Y just in case each X-variable is either also in Y or is a causal source of some Y-variable outside of X. For singleton tuples, ⟨x⟩ is t-prior to ⟨y⟩ iff either x → y or x = y, where → is the causal-source relation on single variables as before.

Somewhat surprisingly--since this is not at all evident in the definition-- t-priority (and by duality b-priority) turns out to be transitive, anti-symmetric relative just to ≐, and is hence a classical partial order.

Proof. For anti-symmetry, observe that if X is t-prior to Y and conversely, then X-not-Y t-precedes Y-not-X and conversely--which by the irreflexivity of t-precedence holds only if X-not-Y and Y-not-X are both null, i.e., only if X ≐ Y. To show transitivity, assume that X is t-prior to Y, that Y is t-prior to Z, and take X', Y', Z' to be the subtuples respectively of X, Y, Z formed by deleting just the variables common to all three of X,Y,Z. Then X' is t-prior to Y' and Y' is t-prior to Z' (since deleting some or all of the variables common to two tuples does not alter whether one is t-prior to the other.) We next observe that if some variable x in X'-not-Z' were not to be a source of any variable in Z'-not-X', x would start an arbitrarily long sequence $x \dashrightarrow y \dashrightarrow z = x' \dashrightarrow y' \dashrightarrow z' = x'' \dashrightarrow y'' \dashrightarrow z'' = \ldots$ of variables cyclically from X',Y',Z' in which $\dashrightarrow$ is variously either identity or the causal-source relation and must be the latter at least once in each cycle, so that the arrows in the entailed subsequence $x \rightarrow z \rightarrow z'' \ldots$ are all causal. (This is because by definition of t-priority, x must either be the same as or a source of some y in Y' which in turn must be the same as or a source of some z in Z', whence $x \rightarrow z$ since $x = y = z$ is precluded by construction of X',Y',Z'. And z must be in X', since otherwise it would be in Z'-not-X' contrary to hypothesis. Similarly, $z \rightarrow z'$ for some z' in Z' which must also be in X' if z' is not to violate the assumption that x is a source of no variable in Z'-not-X'; and so on.) Since tuple Z' is finite, this sequence would eventually violate the causal-source relation's transitivity and irreflexivity. Hence X' must be t-prior to Z', and restoring the deleted variables in common yields that X is t-prior to Z. □

Cleansing s-determination of irrelevancies by combining it with t-priority yields the order properties that we seek. Specifically,

Definition 2.14. A tuple X t(ightly) determines tuple Y--symbolized $X \Rightarrow Y$ -- iff X both s-determines Y and is t-prior to Y.

It is obvious but worth mention that if X s-determines Y, then some subtuple X' of X t-determines Y. (Proof: Let X' comprise just the variables in X that are either in Y or are a source of some variable in Y-not-X. Then X', too, s-determines Y and by construction is also t-prior to Y.) Note also that s-determination, t-priority, and hence t-determination are all vacuously reflexive.

Since s-determination and t-priority are both transitive, so is t-determination; and the classic anti-symmetry of t-priority makes t-determination also classically anti-symmetric. Hence t-determination is a classical partial order. This means that if a t-determination series is any sequence $\ldots \Rightarrow \underset{\wedge}{X}_i \Rightarrow \underset{\wedge}{X}_{i+1} \Rightarrow \underset{\wedge}{X}_{i+2} \Rightarrow \ldots$ of tuples of variables in which each $\underset{\wedge}{X}_i$ t-determines $\underset{\wedge}{X}_{i+1}$ and does not contain exactly the same variables as $\underset{\wedge}{X}_{i+1}$, no t-determination series ever makes a loop.

As background for future macrostructural studies, it may be worthwhile to put the main combinatorial properties of these relations on record:

<u>Theorem</u> 11: For any Tuples: (1) $\langle \underset{\wedge}{X}, \underset{\wedge}{Z} \rangle$ t-precedes $\underset{\wedge}{Y}$ iff $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$ both t-precede $\underset{\wedge}{Y}$ separately. (2) If $\underset{\wedge}{X}_1$ b-precedes, or t-precedes, or s-determines $\underset{\wedge}{Y}_1$, and $\underset{\wedge}{X}_2$ correspondingly b-precedes, or t-precedes, or s-determines $\underset{\wedge}{Y}_2$, then $\langle \underset{\wedge}{X}_1, \underset{\wedge}{X}_2 \rangle$ respectively b-precedes, t-precedes, or s-determines $\langle \underset{\wedge}{Y}_1, \underset{\wedge}{Y}_2 \rangle$. (3) If $\underset{\wedge}{X}$ t-precedes $\underset{\wedge}{Y}$, or is t-prior to $\underset{\wedge}{Y}$, then each subtuple $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}_1$ of $\underset{\wedge}{X}$ respectively t-precedes or is t-prior to every supertuple $\langle \underset{\wedge}{Y}, \underset{\wedge}{Z}_2 \rangle$ of $\underset{\wedge}{Y}$. (4) If $\underset{\wedge}{X}$ is t-prior to $\underset{\wedge}{Y}$ and no variable in $\underset{\wedge}{Z}$ is in $\underset{\wedge}{Y}$ unless it is also in $\underset{\wedge}{X}$, then $\langle \underset{\wedge}{X}, \underset{\wedge}{Z} \rangle$ is t-prior to $\langle \underset{\wedge}{Y}, \underset{\wedge}{Z} \rangle$. (<u>Corollary</u>. If $\underset{\wedge}{X}$ t-determines $\underset{\wedge}{Y}$ and no variable in $\underset{\wedge}{Z}$ is in $\underset{\wedge}{Y}$ unless it is also in $\underset{\wedge}{X}$, then $\langle \underset{\wedge}{X}, \underset{\wedge}{Z} \rangle$ t-determines $\langle \underset{\wedge}{Y}, \underset{\wedge}{Z} \rangle$.) (4') $\langle \underset{\wedge}{X}, \underset{\wedge}{Z} \rangle$ is t-prior to $\langle \underset{\wedge}{Y}, \underset{\wedge}{Z} \rangle$ iff $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$ is t-prior to $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$.) (5) If $\underset{\wedge}{X}$ and $\underset{\wedge}{z}$ both t-precede $\underset{\wedge}{Y}$, then $\underset{\wedge}{X}$ t-precedes $\underset{\wedge}{Y}$-not-$\underset{\wedge}{z}$. (5') If $\langle \underset{\wedge}{X}, \underset{\wedge}{Z} \rangle$ t-precedes $\underset{\wedge}{Y}$, then $\underset{\wedge}{X}$ t-precedes $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$. (6) If $\underset{\wedge}{X}$ t-precedes $\underset{\wedge}{Y}$, then $\underset{\wedge}{X}$ is t-prior to $\underset{\wedge}{Y}$. (7) if $\underset{\wedge}{X}\underline{R}\underset{\wedge}{Y}$ for any of the relations $\underline{R}$ defined here in terms of b- or t-precedence and causal determination, while $\underset{\wedge}{X}' \doteq \underset{\wedge}{X}$ and $\underset{\wedge}{Y}' \doteq \underset{\wedge}{Y}$, then also $\underset{\wedge}{X}'\underline{R}\underset{\wedge}{Y}'$.

<u>Proofs</u>. (1) and (2) are obvious, and (7) even more so. (3) is immediate for t-precedence, and from there for t-priority by noting that ($\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}_1$)-not-$\langle \underset{\wedge}{Y}, \underset{\wedge}{Z}_2 \rangle$ is a subtuple of $\underset{\wedge}{X}$-not-$\underset{\wedge}{Y}$ while $\underset{\wedge}{Y}$-not-$\underset{\wedge}{X}$ is a subtuple of $\langle \underset{\wedge}{Y}, \underset{\wedge}{Z}_2 \rangle$-not-($\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}_1$)). (4) holds because under the stipulated conditions, $\langle \underset{\wedge}{X}, \underset{\wedge}{Z} \rangle$-not-$\langle \underset{\wedge}{Y}, \underset{\wedge}{Z} \rangle$ = $\underset{\wedge}{Y}$-not-$\underset{\wedge}{X}$; the corollary is obvious under the definitions of t- and s-determination. For (4'), since $\langle \underset{\wedge}{X}, \underset{\wedge}{Z} \rangle$-not-$\langle \underset{\wedge}{Y}, \underset{\wedge}{Z} \rangle$ = $\underset{\wedge}{X}$-not-$\langle \underset{\wedge}{Y}, \underset{\wedge}{Z} \rangle$ = ($\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$)-not-($\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$), $\langle \underset{\wedge}{X}, \underset{\wedge}{Z} \rangle$ is t-prior to $\langle \underset{\wedge}{Y}, \underset{\wedge}{Z} \rangle$ iff $\langle \underset{\wedge}{X}, \underset{\wedge}{Z} \rangle$-not-$\langle \underset{\wedge}{Y}, \underset{\wedge}{Z} \rangle$ t-precedes $\langle Y, Z \rangle$-not-$\langle X, Z \rangle$ iff ($\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$)-not-($\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$) t-precedes ($\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$)-not-($\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$) iff $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$ is t-prior to $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$. In (5), if $\underset{\wedge}{X}$ and $\underset{\wedge}{z}$ are as stipulated, $\underset{\wedge}{z} \rightarrow \underset{\wedge}{y}^*$ for some $\underset{\wedge}{y}^*$ in $\underset{\wedge}{Y}$ and hence (since $\underset{\wedge}{y}^* \neq \underset{\wedge}{z}$) in $\underset{\wedge}{Y}$-not-$\underset{\wedge}{z}$, while for each $\underset{\wedge}{x}$ in $\underset{\wedge}{X}$, $\underset{\wedge}{x} \rightarrow \underset{\wedge}{y}'$ for some $\underset{\wedge}{y}'$ in $\underset{\wedge}{Y}$. Either

this y' is in Y-not-z (if $z \neq y'$), or $x \rightarrow z \rightarrow y^*$ (if $z = y'$) in which case x is a source of y* in Y-not-z. For (5'), assume that ⟨X,Z⟩ t-precedes Y, i.e., by (1), that X and $Z = \langle z_1, \ldots, z_m \rangle$ both t-precede Y. Then z_m also t-precedes Y (cf. (1)), so by (5) and (3), X and $\langle z_1, \ldots, z_{m-1} \rangle$ both t-precede Y-not-z_m. Induction on *m* thus concludes that X t-precedes Y-not-Z. For (6), assume that X t-precedes Y and let Z comprise just the variables common to X and Y. Then ⟨X-not-Z,Z⟩ t-precedes Y and, by (1) so does Z; hence by (5'), X t-precedes Y-not-Z. But Y-not-Z = Y-not-X by definition of Z, so X and, by (3), also X-not-Y t-precedes Y-not-X. That is, X is t-prior to Y. □

<u>The interpretive character of t-determination.</u>

Using the principles listed in Theorem 11, the macrostructural nature of t-determination can be explicated as

<u>Theorem</u> 12. For any tuples X and Y, X t-determines Y just in case, for some positive integer *n*+1, there exist tuples $X_1, \ldots, X_n, X_{n+1}, Y_1, \ldots, Y_n, Z$ (any of which can be null) such that (*a*) $X \doteq \langle X_1, \ldots, X_n, X_{n+1}, Z \rangle$ and $Y \doteq \langle Y_1, \ldots, Y_n, Z \rangle$; (*b*) for each *i* = 1,...,*n*, X_i is a strictly complete source of each variable in Y_i; (*c*) X_{n+1} t-precedes $\langle Y_1, \ldots, Y_n \rangle$; and (*d*) every variable in X-not-Z that is a source of some Y-variable in *I*(X) also t-precedes $\langle Y_1, \ldots, Y_n \rangle$-not-*I*(X). (<u>Note</u>: If X's interior is null or disjoint from Y, condition (*d*) is vacuous.)

We shall not bother to prove Theorem 12 here, for the argument is reasonably routine and only brief heuristic use will be made of this result here. But when we have envisioned a structure of macrocausal connections among the variables in tuples X and Y, Theorem 12 makes it easy for us to appraise whether X t-determines Y.

With only marginal exceptions, whenever X not merely s-determines Y but is intuitively fully antecedent to it, X also t-determines Y. The exceptions are certain cases that violate condition (*c*) or (*d*) in Theorem 12. The simplest example of (*d*)-failure is the relation between X = ⟨x,y⟩ and Y = ⟨y⟩ when x is a strictly complete source of y. Here X s-determines Y but cannot t-determine it insomuch as Y-not-X is null but X-not-Y is not. (This example's violation of (d) in Theorem 12 usefully illustrates the force of that condition.) And cases where X seems fully antecedent to Y without satisfying (c) are illustrated by $X = \langle x_1, x_2, z\rangle$, $Y = \langle y, z\rangle$, when x_1 is a strictly complete source of y, and x_2 is a source of z but not of y. Here again X s-determines Y, and moreover every variable in X is either in Y or is a source of some variable in Y; yet X does not t-determine Y because X-not-Y ($= \langle x_1, x_2\rangle$) does not t-precede Y-not-X ($= \langle y\rangle$).

Even so, t-determination generally excels at the finer macrocausal order distinctions missed by s-determination. One test, it will be recalled, is the asymmetry between X and $\langle X,Y\rangle$ when X is a strictly complete source of each variable in Y. Application of Theorem 12 shows that X t-determines $\langle X,Y\rangle$ in this case, while by t-determination's classical anti-symmetry and the preclusion of $X \doteq \langle X,Y\rangle$ in this case, $\langle X,Y\rangle$ does not t-determine X. And if X is a strictly complete source of both Z and Y, while each Z-variable is also a source of some variable in Y--our other test case-- $\langle X,Z\rangle$ t-determines $\langle X,Y\rangle$ but not conversely.

Theorem 12 can easily generalize upon these special cases of t-determination. But more fundamental is that if Z is any tuple that interests us, say because it t-determines output tuple Y, and some subtuple Z' of Z is in turn t-determined by some tuple X of more remote Y-sources, then $\langle$Z-not-Z',X$\rangle$ t-determines Y. (Proof: By Theorem 11-4 Corollary, if X t-determines Z', then $\langle$X,Z-not-Z'$\rangle$ t-determines $\langle$Z',Z-not-Z'$\rangle \doteq$ Z and hence also, by transitivity of t-determination, any Y that Z in turn t-determines.) This means in particular that starting with a given output tuple Y, if $\ldots \Rightarrow Z_{k-1} \Rightarrow Z_k \Rightarrow Z_{k+1} \Rightarrow \ldots \Rightarrow Y$ is a precession of inclusively complete Y-sources in which each Z_{k-1} is obtained from Z_k by replacing one or more variables in Z_k by strictly complete sources thereof, this sequence is a t-determination series, leading to Y, in which each Z_{k-1} t-determines Z_{k+1} and hence all subsequent tuples in the sequence through mediation by Z_k. (We shall later examine causal compositions for such t-determination sequences in some detail.) Accordingly, it would appear that all macrostructural mediation relations of interest to multivariate analysis are contained in the classical partial order of t-determination.

Actually, the causal-determination relation that proves to be most powerful for macrocausal analysis is not bare t-determination in the absolute sense of Def. 2.14, but a relativizing of this to the microcausal path structure within a particular background tuple $\underset{\wedge}{W}$ that includes all the variables whose causal connections are explicitly at issue. To define relativized t-determination and establish the theorems applying thereto, we need merely construe all references to causal-source connections among single variables in Def. 2.11 <u>et seq</u>. to bear the implicit qualification "relative to $\underset{\wedge}{W}$" with the understanding that $\underset{\wedge}{x}$ is a source of $\underset{\wedge}{y}$ relative to $\underset{\wedge}{W}$ iff there is a path from $\underset{\wedge}{x}$ to $\underset{\wedge}{y}$ within $\underset{\wedge}{W}$ in the sense of Def. 2.7. With fixed $\underset{\wedge}{W}$, this relativising to $\underset{\wedge}{W}$ of source-relation $\twoheadrightarrow$ does not alter its strict-partial-order character, so all definitions and theorems previously developed in terms of $\twoheadrightarrow$ follow exactly as before except that these, too, are now generally relative to $\underset{\wedge}{W}$. In some cases, there is no essential difference between a relation or principle based on absolute $\twoheadrightarrow$ and its relativized counterpart. In particular, for any subtuples $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$ of $\underset{\wedge}{W}$, $\underset{\wedge}{X}$ s-determines $\underset{\wedge}{Y}$ relative to $\underset{\wedge}{W}$ just in case $\underset{\wedge}{X}$ s-determines $\underset{\wedge}{Y}$ absolutely. So when $\underset{\wedge}{W}$ contains all variables at issue, the only difference between $\underset{\wedge}{X}$ t-determining $\underset{\wedge}{Y}$ absolutely and doing so relative to $\underset{\wedge}{W}$ is a strengthening of t-precedence requirements (<u>c</u>) and (<u>d</u>) in Theorem 12 to t-precedence relative to $\underset{\wedge}{W}$.

<u>Macrostructural mediation</u>.

According to our introduction, the theory of causal macrostructure aspires to develop an account of causal connections among groups of variables that parallels the logic of microcausal path structure. Before seeking to fulfil that promise, however, we had best make clear just what information a path digraph does express.

Were there nothing more to microcausal path structure than a partial ordering of causation among single variables, any of the partial-order relations on Tuples already identified here would be a macrocausal parallel. But microstructural path digraphs say a great deal more than that--enough, in fact, to warrant a list:

<u>What causal-path digraphs represent.</u>

1. Causal-source connections.
2. Causal mediation.
3. Causal disconnection.
4. Causal determination.
5. Causal composition.

1. That the microstructural path digraph for a tuple $\underset{\wedge}{X}$--for convenient reference call this structure π_X--expresses binary causal-source relations within $\underset{\wedge}{X}$ by the directed lines connecting some $\underset{\wedge}{X}$-variables to others is the most conspicuous feature of π_X. But what the arrows in π_X stand for is not <u>merely</u> the causal-source relation $\longrightarrow$ but a very special instance of this relative to $\underset{\wedge}{X}$. A path from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ in π_X indeed conveys that $\underset{\wedge}{x}_i \rightarrow \underset{\wedge}{x}_j$; but a multiplicity of π_X-paths from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ has a structural significance that it could not have were this just a way to express $\underset{\wedge}{x}_i \rightarrow \underset{\wedge}{x}_j$, nor does lack of path from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ in π_X imply, conversely, that $\underset{\wedge}{x}_i$ is not a source of $\underset{\wedge}{x}_j$. The absence of particular path connections in π_X is not just an arbitrary omission of source relations that we choose to disregard, but is fully as essential to what π_X tells us as are the paths that π_X does contain.

2. Similar remarks apply to π_X's representation of mediation among the variables in tuple $\underset{\wedge}{X}$. Manifestly, if a π_X-path from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ passes through $\underset{\wedge}{x}_k$, then $\underset{\wedge}{x}_i$ influences $\underset{\wedge}{x}_j$ through the mediation of $\underset{\wedge}{x}_k$. But failure of π_X to contain a path from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ through $\underset{\wedge}{x}_k$ does not say, conversely, that $\underset{\wedge}{x}_k$ does not mediate between $\underset{\wedge}{x}_i$ and $\underset{\wedge}{x}_j$--it is entirely possible for $\underset{\wedge}{x}_i$ to be a source of $\underset{\wedge}{x}_k$, and $\underset{\wedge}{x}_k$ of $\underset{\wedge}{x}_j$, without these connections being featured in π_X. And a multiplicity of paths from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{x}_j$ with some but not all passing through $\underset{\wedge}{x}_k$ says far more about the causal relations among these variables than just $\underset{\wedge}{x}_i \rightarrow \underset{\wedge}{x}_k \rightarrow \underset{\wedge}{x}_j$.

3. How path digraph π_X also represents disconnection (total mediation) is explained in Theorem 3 (p. 2.14). Not all disconnection possibilities among $\underset{\wedge}{X}$-variables are ajudicated by π_X. But if $\underset{\wedge}{x}_j$ is in $\underline{I}(\underset{\wedge}{X})$ and $\underset{\wedge}{X}_k$ contains at least one variable on

each total path to $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$, then for any other variable $\underset{\wedge}{x}_i$ in $\underset{\wedge}{X}$, π_X reveals whether $\underset{\wedge}{X}_k$ disconnects $\underset{\wedge}{x}_i$ from $\underset{\wedge}{x}_j$, namely, No if $\underset{\wedge}{x}_i$ is in the $\underset{\wedge}{X}$-buffer from $\underset{\wedge}{X}_k$ to $\underset{\wedge}{x}_j$, and Yes otherwise. π_X's expression of disconnection depends as importantly on which $\underset{\wedge}{X}$-variables it does <u>not</u> link by paths as on those it does, and is where the deeper significance of path structure begins to emerge. Even so, the abstract definition of total mediation in terms of path connections manifests little reason to prize this information for its own sake. Rather, disconnection's payoff is its import for causal composition (cf. Theorem 7, p. 2.29).

4 & 5. Most fundamentally, path digraph π_X identifies which subtuples of $\underset{\wedge}{X}$ are complete sources of what other X-variables (cf. Theorem 4, p. 2.16), <u>and</u> which of the strict/extended causal regularities that govern these determinations derive from which others by compositions of transducers and subtuple selectors (cf. Theorems 5 & 7; more comprehensively, see Theorems 15 & 24, below). This is where lies the ultimate challenge for causal analysis: to identify the parameters of (relatively) basic causal mechanisms from which are composed the overarching causal behaviors of more complex natural systems. The logic of causal explanation is multi-leveled: Not merely do variables (more precisely, instantiations of their values) cause one another according to lawful regularities, but these laws themselves are generally the way they are as a logical consequence of more fundamental laws. <u>That</u> is what makes partial/total mediation so central for causality, and what it is that path digraphs most deeply represent.

It is evident from this review that no partial ordering of Tuples properly qualifies as a macrocausal counterpart of path structure unless it carries information about disconnection and causal composition as well as causal determination. We have looked with some care at the causal-determination ordering of Tuples, but have said nothing as yet about macromediation. The central concept needed for this--macro-disconnection--is just micro-disconnection writ large, namely,

<u>Definition</u> 2.15. Tuple $\underset{\wedge}{Z}$ (<u>macrostructurally</u>) <u>disconnects</u> tuple $\underset{\wedge}{X}$ from tuple $\underset{\wedge}{Y}$ iff $\underset{\wedge}{Z}$ microstructurally disconnects each variable in $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$ from every variable

in $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$. Tuple $\underset{\wedge}{Z}$ <u>properly disconnects</u> $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$ iff $\underset{\wedge}{Z}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$ and neither $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$ nor $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$ is null.

Note that $\underset{\wedge}{Z}$ cannot disconnect $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$ unless all variables common to $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$ are also in $\underset{\wedge}{Z}$, since $\underset{\wedge}{Z}$ fails to micro-disconnect $\underset{\wedge}{x}_i$ in $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$ from $\underset{\wedge}{y}_j$ in $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$ if $\underset{\wedge}{x}_i = \underset{\wedge}{y}_j$. (Cf. Def. 2.8. This point will prove critical later.) Also worth making explicit is

<u>Theorem</u> 13. (1) If $\underset{\wedge}{X} \doteq \langle \underset{\wedge}{X}_1, \ldots, \underset{\wedge}{X}_m \rangle$ and $\underset{\wedge}{Y} \doteq \langle \underset{\wedge}{Y}_1, \ldots, \underset{\wedge}{Y}_n \rangle$, $\underset{\wedge}{Z}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$ just in case $\underset{\wedge}{Z}$ disconnects each $\underset{\wedge}{X}_i$ ($\underline{i} = 1, \ldots, \underline{m}$) from each $\underset{\wedge}{Y}_j$ ($\underline{j} = 1, \ldots, \underline{n}$). <u>Corollary</u>. $\underset{\wedge}{Z}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$ just in case $\underset{\wedge}{Z}$ disconnects each subtuple of $\underset{\wedge}{X}$ from each subtuple of $\underset{\wedge}{Y}$. (2) $\underset{\wedge}{Z}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$ just in case $\underset{\wedge}{Z}$ disconnects $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$ from $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$. <u>Corollary</u>. $\underset{\wedge}{Z}$ disconnects each subtuple of itself from every $\underset{\wedge}{Y}$, and every $\underset{\wedge}{X}$ from each subtuple of itself.

Both parts of Theorem 13 are immediate from Def. 2.15.

An intuitive anomaly under Def. 2.15 is that every Tuple disconnects itself from itself. But if "$\underset{\wedge}{Z}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$" is understood as elliptic for "$\underset{\wedge}{Z}$ disconnects the non-$\underset{\wedge}{Z}$ part of $\underset{\wedge}{X}$ from the non-$\underset{\wedge}{Z}$ part of $\underset{\wedge}{Y}$," the discomfort vanishes except for the residual awkwardness that any singleton tuple $\langle \underset{\wedge}{x} \rangle$ macrostructurally disconnects $\langle \underset{\wedge}{x} \rangle$ from $\langle \underset{\wedge}{x} \rangle$ even though it does not disconnect $\underset{\wedge}{x}$ from $\underset{\wedge}{x}$ microstructurally (Def. 2.8). Proper disconnection avoids this peculiarity--i.e., no Tuple properly disconnects any subtuple of itself from any subtuple of itself. But for most technical purposes, the non-nullity condition in proper disconnection is a distracting irrelevancy.

When coupled with determination, macrostructural disconnection is finitely identifiable in terms of microcausal path structure as

<u>Theorem</u> 14. If tuple $\underset{\wedge}{Z}$ s-determines tuple $\underset{\wedge}{Y}$, then for any tuple $\underset{\wedge}{X}$, $\underset{\wedge}{Z}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$ just in case ($\underline{a}$) $\underset{\wedge}{X}$-not-$\underset{\wedge}{Z}$ and $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$ are disjoint (i.e., every variable common to $\underset{\wedge}{X}$ and $\underset{\wedge}{Y}$ is also in $\underset{\wedge}{Z}$), and ($\underline{b}$) every path in $\langle \underset{\wedge}{X}, \underset{\wedge}{Z}, \underset{\wedge}{Y} \rangle$ from

any variable in X-not-Z to some variable in Y-not-Z passes through Z. Corollary. The theorem also holds if clause (b) is replaced by: (b') every path in $\langle X,Z,Y\rangle$ from any variable in X to some variable in Y passes through Z.

Proof. The theorem holds vacuously if either X-not-Z or Y-not-Z is null. Otherwise, let x_i be any variable in X-not-Z and y_j any variable in Y-not-Z. Since $Z \stackrel{s}{\Rightarrow} Y$ by stipulation, y_j is interior to $\langle Z,Y\rangle$ with all total paths to y_j therein beginning with a variable in $E(Z)$; hence all total paths to y_j in $\langle X,Z,Y\rangle$ pass through Z. So by Theorem 3, if $x_i \neq y_j$ with y_j interior to $\langle X,Z,Y\rangle$ while all paths from x_i to y_j in $\langle X,Z,Y\rangle$ pass through Z, then Z disconnects X from Y. Conversely, it is immediate from Def. 2.8 that Z fails to disconnect x_i from y_j either if $x_i = y_j$ or if there is a path from x_i to y_j in $\langle X,Z,Y\rangle$ that does not pass through Z. The corollary follows by observation that (b') is equivalent to (b), insomuch as a path from X to Y in $\langle X,Z,Y\rangle$ that begins with a variable common to X and Z, or ends with one common to Z and Y, thereby passes through (i.e. contains a variable in) Z. □

When X s-determines Y, we can combine the inclusive causal regularities $\{y_i = \phi_i(X)\}$ by which X determines the single variables $\{y_i\}$ in Y-not-X, together with noncausal identity-selector functions that pick out of X the Y-variables also in X, into a single macrostructural quasi-causal regularity $Y = \phi(X)$ defined as follows:

Definition 2.16. Tuple $X = \langle x_1,\ldots,x_m\rangle$ determines tuple $Y = \langle y_1,\ldots,y_n\rangle$ under quasi-causal regularity $Y = \phi(X)$ iff (a) ϕ is a function from the logical range of X into the cartesian product of the ranges of the variables in Y so that $\phi(X) = \langle \phi_1(X),\ldots,\phi_m(X)\rangle$; (b) for each Y-variable y_i in Y-not-X, $y_i = \phi_i(X)$ is an inclusive causal regularity under which X is an inclusively complete source of y_i; and (c) for any y_j common to Y and X, ϕ_j is the singleton-subtuple-selector function that picks y_j out of X, i.e., if y_j is the kth variable in X, $\phi_j(X) = 0\cdot x_1 + \ldots + 0\cdot x_{k-1} + x_k + 0\cdot x_{k+1} + \ldots + 0\cdot x_m$. (Note: If y_j is also in X, $y_j = \phi_j(X)$ is a noncausal identity-selection of y_j from X even when y_j also has a complete causal source in X.)

In proof of the theorem about to follow, we shall need to speak both of quasi-causal regularities and of the strict causal regularities embedded therein. And since a generalization of this embedding concept will be needed later, we declare

<u>Definition</u> 2.17. (1) If $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ and $\underset{\wedge}{Y}' = \phi'(\underset{\wedge}{X})$ are, ~~or~~ have the form of, quasi-causal regularities in which $\underset{\wedge}{Y} = \langle \underset{\wedge}{y}_1, \ldots, \underset{\wedge}{y}_m \rangle$ and $\underset{\wedge}{Y}' = \langle \underset{\wedge}{y}'_1, \ldots, \underset{\wedge}{y}'_n \rangle$, ~~$\underset{\wedge}{Y}' = \phi'(\underset{\wedge}{X}')$~~ is <u>embedded</u> in $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ iff (<u>a</u>) tuples $\underset{\wedge}{Y}$ and $\underset{\wedge}{X}$ respectively include $\underset{\wedge}{Y}'$ and $\underset{\wedge}{X}'$, and (<u>b</u>) for all <u>i</u> = 1,...,<u>m</u> and <u>j</u> = 1,...,<u>n</u>, if the <u>i</u>th variable $\underset{\wedge}{y}_i$ in $\underset{\wedge}{Y}$ is identical with the <u>j</u>th variable $\underset{\wedge}{y}'_j$ in $\underset{\wedge}{Y}'$, the <u>i</u>th component function $\underset{\wedge}{y}_i = \phi_i(\underset{\wedge}{X})$ in $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ differs from the <u>j</u>th component function $\underset{\wedge}{y}'_j = \phi'_j(\underset{\wedge}{X}')$ in $\underset{\wedge}{Y}' = \phi'(\underset{\wedge}{X}')$ only in containing with null weights the variables in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}'$--i.e., if σ' is the subtuple-selector function that picks $\underset{\wedge}{X}'$ out of $\underset{\wedge}{X}$, there is a permutation operator ρ such that $\phi_i = \phi'_j \rho \sigma'$. [The embedding is <u>pre-emptive</u> iff all variables in $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Y}'$ are in $\underset{\wedge}{X}$, i.e. iff $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ can be identified from $\underset{\wedge}{Y}' = \phi'(\underset{\wedge}{X}')$ just by insertion of null-weight inputs and identity selectors.] (Note that embedding is transitive, i.e., if $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ embeds $\underset{\wedge}{Y}' = \phi'(\underset{\wedge}{X}')$ which in turn embeds $\underset{\wedge}{Y}'' = \phi''(\underset{\wedge}{X}'')$, then $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ embeds $\underset{\wedge}{Y}'' = \phi''(\underset{\wedge}{X}'')$.) (2) A causal regularity $\underset{\wedge}{y}_i = \phi^*_i(\underset{\wedge}{X}^*_i)$ is the <u>proximal core</u> of $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ <u>for</u> $\underset{\wedge}{y}_i$ iff (<u>a</u>) $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ is a quasi-causal regularity, (<u>b</u>) $\underset{\wedge}{y}_i$ is a variable in $\underset{\wedge}{Y}$-not-$\underset{\wedge}{X}$ whose proximal source in $\underset{\wedge}{X}$ is $\underset{\wedge}{X}^*_i$, and (<u>c</u>) $\underset{\wedge}{y}_i = \phi^*_i(\underset{\wedge}{X}^*_i)$ is embedded in $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$. We also say that $\underset{\wedge}{y}_i = \phi^*_i(\underset{\wedge}{X}^*_i)$ is the proximal core of the component $\underset{\wedge}{y}_i = \phi_i(\underset{\wedge}{X})$ of $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ whose output variable is $\underset{\wedge}{y}_i$.

For any quasi-causal regularity $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ and any variable $\underset{\wedge}{y}_i$ in $\underset{\wedge}{Y}$-not-$\underset{\wedge}{X}$ (but not for any $\underset{\wedge}{y}_j$ common to $\underset{\wedge}{Y}$ and $\underset{\wedge}{X}$), there is exactly one regularity $\underset{\wedge}{y}_i = \phi^*_i(\underset{\wedge}{X}^*_i)$ that is the proximal core of $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ for $\underset{\wedge}{y}_i$. Since by stipulation $\underset{\wedge}{X}^*_i$ is the (non-null) proximal source of $\underset{\wedge}{y}_i$ in $\underset{\wedge}{X}$, this $\underset{\wedge}{y}_i = \phi^*_i(\underset{\wedge}{X}^*_i)$ is a strict causal regularity that is proximal within $\langle \underset{\wedge}{X}, \underset{\wedge}{y}_i \rangle$. Evidently, a quasi-causal regularity $\underset{\wedge}{Y}' = \phi'(\underset{\wedge}{X}')$ is embedded in quasi-causal regularity $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ just in case $\underset{\wedge}{X}'$ is in $\underset{\wedge}{X}$, $\underset{\wedge}{Y}'$ is in $\underset{\wedge}{Y}$, and for each variable $\underset{\wedge}{y}'_i$ in $\underset{\wedge}{Y}'$-not-$\underset{\wedge}{X}'$, $\underset{\wedge}{y}'_i$ is not in $\underset{\wedge}{X}$ but has the same proximal source in $\underset{\wedge}{X}'$ as it has in the more inclusive tuple $\underset{\wedge}{X}$.

Now consider the situation wherein $\underset{\wedge}{X}$ s-determines $\underset{\wedge}{Z}$ and $\underset{\wedge}{Z}$ s-determines $\underset{\wedge}{Y}$ under respective quasi-causal regularities $\underset{\wedge}{Z} = \psi(\underset{\wedge}{X})$ and $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z})$, i.e.,

$\langle z_1,\ldots,z_m\rangle = \langle\psi_1(X),\ldots,\psi_m(X)\rangle$ and $\langle y_1,\ldots,y_n\rangle = \langle\phi_1(Z),\ldots,\phi_n(Z)\rangle$ where $Z = \langle z_1,\ldots,z_m\rangle$ and $Y = \langle y_1,\ldots,y_n\rangle$. These macrocausal regularities have a well-defined composition, namely, $Y = \phi\psi(X)$. To appreciate the nature of this formalism, observe that the _i_th component function in $Y = \phi\psi(X)$ is $y_i = \phi_i\psi(X)$, which in turn can be equivalently written in expanded notation as $y_i = \phi_i(\psi_1(X),\ldots,\psi_m(X))$. If the _j_th variable in Z is also in X, say $j = 1$ and $z_1 = x_k$, this is in turn equivalent both to $y_i = \phi_i(z_1,\psi_2(X),\ldots,\psi_m(X))$ and to $y_i = \phi(x_k,\psi_2(X),\ldots,\psi_m(X))$, with similar identity replacements holding for any other Z-variables common to X. Thus $Y = \phi\psi(X)$ efficiently composes into each $y_i = \phi_i(z_1,\ldots,z_m)$ the inclusive causal regularities under which the Z-variables not in X are determined by X.[7] However, we know from discussion of _FT_-1 that the composition of one causal regularity into another does not always preserve causality. So even when $Z = \psi(X)$ and $Y = \phi(Z)$ are both quasi-causal, their composition $Y = \phi\psi(X)$ may not be so. Happily, the conditions under which quasi-causality status is preserved under composition of s-determinations proves to be remarkably simple:

Theorem 15. Let tuple X s-determine tuple Z under quasi-causal regularity $Z = \psi(X)$ while Z in turn s-determines tuple Y under quasi-causal regularity $Y = \phi(Z)$. Then if Z disconnects X from Y, X s-determines Y under quasi-causal regularity $Y = \phi\psi(X)$.

Proof. Assume the theorem's preconditions and for each variable y_j in Y, let $y_j = \phi_j(X)$ be the _j_th component regularity in $Y = \phi(X)$. Then we are to show (_a_) that if y_j is in X, $\phi_j\psi$ is a singleton-subtuple-selector function that picks y_j out of X, whereas (_b_) if y_j is in Y-not-X, $y_j = \phi_j\psi(X)$ is the inclusive causal regularity by which X is an inclusive source of y_j. Case (_a_) is obvious, since by disconnection y_j is then also a variable in Z, say the _k_th. So ϕ_j is a subtuple selector that picks out the _k_th component of its argument (i.e., $\phi_j(Z) = z_k = y_j$), the _k_th component transducer ψ_k in $Z = \psi(X)$ is the subtuple selector that picks z_k, i.e. y_j, out of X, and $\phi_j\psi$ is hence the subtuple selector that picks y_j out of X. In our main case (_b_), either ($\underline{b}_1$) $\underset{\wedge}{y}_j$ is also in $\underset{\wedge}{Z}$ but not in $\underset{\wedge}{X}$, or ($\underline{b}_2$) $\underset{\wedge}{y}_j$ is in neither $\underset{\wedge}{Z}$ nor $\underset{\wedge}{X}$. In case ($\underline{b}_1$), $\underset{\wedge}{y}_j = \phi_j\psi(\underset{\wedge}{X})$ is identical with the $\underline{k}$th component $\underset{\wedge}{z}_k = \psi_k(\underset{\wedge}{X})$ of $\underset{\wedge}{Z} = \psi(\underset{\wedge}{X})$, where $\underset{\wedge}{y}_j$ is the $\underline{k}$th variable in $\underset{\wedge}{Z}$. But $\underset{\wedge}{z}_k = \psi_k(\underset{\wedge}{X})$ is the inclusive causal regularity under which $\underset{\wedge}{X}$ is an inclusively complete source of $\underset{\wedge}{z}_k$ and hence by identity so is $\underset{\wedge}{y}_j = \phi_i\psi(\underset{\wedge}{X})$. Finally, for case ($\underline{b}_2$), let $\underset{\wedge}{y}_j = \phi^*_j(\underset{\wedge}{Z}^*_j)$ be the proximal core of $\underset{\wedge}{Y} = \phi(\underset{\wedge}{X})$ for $\underset{\wedge}{y}_j$, and presume without essential loss of generality that $\underset{\wedge}{Z}$ has been so permuted that $\underset{\wedge}{Z} = \langle \underset{\wedge}{Z}^*_j, \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}^*_j\rangle$ and $\underset{\wedge}{Z}^*_j = \langle \underset{\wedge}{X}_j, \underset{\wedge}{Z}^!_j\rangle$ where $\underset{\wedge}{X}_j$ comprises just the $\underline{r}$ variables ($\underline{r} \geq 0$) common to $\underset{\wedge}{Z}^*_j$ and $\underset{\wedge}{X}$, and $\underset{\wedge}{Z}^!_j$ (= $\underset{\wedge}{Z}^*_j\text{-not-}\underset{\wedge}{X}_j$), if not null, is causally well-ordered. (There does not generally exist a permutation of $\underset{\wedge}{Z}$ that achieves these constraints simultaneously for all variables $\underset{\wedge}{y}_j$ in $\underset{\wedge}{Y}\text{-not-}\underset{\wedge}{Z}$; but we are dealing with just one arbitrarily selected $\underset{\wedge}{y}_j$ therein and leaving suppressed the permutation that would have to be made explicit if the present proof were to be given in complete detail.) Then $\underset{\wedge}{y}_j = \phi^*_j(\underset{\wedge}{Z}^*_j)$ is identical with $\underset{\wedge}{y}_j = \phi^*_j(\underset{\wedge}{X}_j, \underset{\wedge}{Z}^!_j)$ where $\underset{\wedge}{X}_j$ comprises the first $\underline{r}$ variables in $\underset{\wedge}{Z}$ and $\underset{\wedge}{Z}^!_j$ is either null or consists of $\underset{\wedge}{Y}$-variables $\langle \underset{\wedge}{y}_{r+1}, \ldots, \underset{\wedge}{y}_{r+m}\rangle$ for some $\underline{m} \geq 1$. Now, regardless of whether $\underset{\wedge}{Z}^!_j$ is null, $\underset{\wedge}{y}_j = \phi^*_j(\underset{\wedge}{X}_j, \underset{\wedge}{Z}^!_j)$ is proximal not only in $\langle \underset{\wedge}{Z}, \underset{\wedge}{y}_j\rangle$ but also, by our disconnection premise, in $\underset{\wedge}{W}_j =_{def} \langle X, Z, y_j\rangle$. [Reason: $\underset{\wedge}{y}_j$ has a non-null proximal source $\underset{\wedge}{W}^*_j$ in $\underset{\wedge}{W}_j$, while by assumption, subtuple $\langle \underset{\wedge}{X}_j, \underset{\wedge}{Z}^!_j\rangle$ of $\underset{\wedge}{Z}$ is the proximal source of $\underset{\wedge}{y}_j$ in $\langle \underset{\wedge}{Z}, \underset{\wedge}{y}_j\rangle = \underset{\wedge}{W}_j\text{-not-}\underset{\wedge}{X}_0$ where $\underset{\wedge}{X}_0 =_{def} \underset{\wedge}{X}\text{-not-}\underset{\wedge}{Z}$. If some variable $\underset{\wedge}{x}_0$ in $\underset{\wedge}{X}_0$ were to be in $\underset{\wedge}{W}^*_j$, then there would be a path from $\underset{\wedge}{x}_0$ to $\underset{\wedge}{y}_j$ in $\underset{\wedge}{W}_j$ that does not pass through Z, contrary under Theorem 14 to stipulation that $\underset{\wedge}{Z}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$. Hence $\underset{\wedge}{y}_j$'s proximal source in $\underset{\wedge}{W}_j$ must also be its proximal source in $\underset{\wedge}{W}_j\text{-not-}\underset{\wedge}{X}_0 = \langle \underset{\wedge}{Z}, \underset{\wedge}{y}_j\rangle$.] If $\underset{\wedge}{Z}^!_j$ is null, $\underset{\wedge}{y}_j = \phi^*_j(\underset{\wedge}{X}_j, \underset{\wedge}{Z}^!_j)$ simplifies to $\underset{\wedge}{y}_j = \phi^*_j(\underset{\wedge}{X}_j)$; and by considering how the relevant subtuple selector picks $\underset{\wedge}{X}_j$ out of $\underset{\wedge}{Z}$ in $\underset{\wedge}{y}_j = \phi_j(\underset{\wedge}{Z})$ and accordingly gives non-null weight in $\phi_j\psi(\underset{\wedge}{X})$ just to the components of $\psi(\underset{\wedge}{X})$ that are variables in $\underset{\wedge}{X}_j$ picked out of $\underset{\wedge}{X}$ by the noncausal identity-selector components of ψ, we can see that $\underset{\wedge}{y}_j = \phi^*_j(\underset{\wedge}{X}_j)$ is embedded in $\underset{\wedge}{y}_j = \phi_j\psi(\underset{\wedge}{X})$. Hence $\underset{\wedge}{y}_j = \phi_j\psi(\underset{\wedge}{X})$ is the inclusive regularity under which $\underset{\wedge}{X}$ determines $\underset{\wedge}{y}_j$ in this null-$\underset{\wedge}{Z}^!_j$ subcase, as was to be shown. Alternatively, if $\underset{\wedge}{Z}^!_j$ in $\underset{\wedge}{y}_j = \phi^*_j(\underset{\wedge}{Z}^*_j) = \phi^*_j(\underset{\wedge}{X}_j, \underset{\wedge}{Z}^!_j)$ is not null, we have $\underset{\wedge}{Z}^!_j = \langle \underset{\wedge}{z}_{r+1}, \ldots, \underset{\wedge}{z}_{r+m}\rangle$ for some $\underline{m} \geq 1$, with the causal well-ordering stipulated for Z_j' entailing that there is no path in $W_j =_{def} \langle X, Z, y_j \rangle$ from any z_{r+i} in Z_j' to any of the variables $z_{r+1}, \ldots, z_{r+i-1}$ that precede z_{r+i} in Z_j'. The proximal core $y_j = \phi_j^*(X_j, z_{r+1}, \ldots, z_{r+m})$ of $y_j = \phi_j(Z)$ is proximal not only in $\langle Z, y_j \rangle$ but also in W_j $(= \langle X, Z, y_j \rangle)$ for the reason already given. And if $z_{r+1} = \psi_{r+1}^*(X_{r+1}^*)$ is the proximal core of $Z = \psi(X)$ for z_{r+1}, $z_{r+1} = \psi_{r+1}^*(X_{r+1}^*)$ is proximal not only in $\langle X, z_{r+1} \rangle$ but also in W_j, insomuch as none of variables $\langle z_{r+2}, \ldots, z_{r+m}, y_j \rangle$ is a source of z_{r+1}. So by <u>CmP</u>-4, $y_j = \phi_j^*(X_j, \psi_{r+1}^*(X_{r+1}^*), z_{r+2}, \ldots, z_{r+m})$ is a strict causal regularity that is proximal in W_j-not-z_{r+1}. Moreover, the proximal core $z_{r+2} = \psi_{r+2}^*(X_{r+2}^*)$ of $Z = \psi(X)$ for z_{r+2} is proximal not only in $\langle X, z_{r+2} \rangle$ but also in W_j-not-z_{r+1}, since none of variables $\langle z_{r+3}, \ldots, z_{r+m}, y_j \rangle$ is a source of z_{r+2}. So again by <u>CmP</u>-4, $y_j = \phi_j^*(X_j, \psi_{r+1}^*(X_{r+1}^*), \psi_{r+2}^*(X_{r+2}^*), z_{r+3}, \ldots, z_{r+m})$ is a strict causal regularity that is proximal in W_j-not-$\langle z_{r+1}, z_{r+2} \rangle$. Continuing in this way eventually shows (technically, by induction) that $y_j = \phi_j^*(X_j, \psi_{r+1}^*(X_{r+1}^*), \ldots, \psi_{r+m}^*(X_{r+m}^*))$ is a proximal regularity in W_j-not-$Z_j' = \langle X, y_j \rangle$. And the latter regularity is embedded in $y_j = \phi_j \psi(X)$. (Recall that by our stipulated ordering of Z, $\phi_j(Z) = \phi_j^*(Z_j^*) + 0 \cdot Z$-not-$Z_j^*$ with $Z_j^* = \langle X_j, z_{r+1}, \ldots, z_{r+m} \rangle$, while the first <u>r</u> components of $Z = \phi(X)$ are identity-selections of the variables X_j common to Z_j^* and X.) So $y_j = \phi_j/(X)$ is the inclusive causal regularity under which X is an inclusively complete source of y_j. □

Much of Theorem 15's proof consists of struggle with trivial but obfuscating technicalities concerning null weights and noncausal identity selections. But at the theorem's heart lies an argument that is neither trivial nor obvious. To appreciate how this result is surprising yet true, a gratifying macrostructural tidiness in what might well have turned out to be an intractable snarl of microcausal proximalities, it is helpful to re-trace the theorem's proof in its special instance wherein Y is a singleton $\langle y \rangle$, and X is a strictly complete source of each variable in $Z = \langle z_1, \ldots, z_m \rangle$ (whence also X and Z are disjoint). In this simplified case, if $y = \phi(z_1, \ldots, y_m)$ and $\{z_i = \psi_i(X)\}$ are all strictly causal with Z disconnecting X from y, it is evident from <u>CmP</u>-4 that $y = \phi(\psi_1(X), \ldots, \psi_m(X))$ would be causal if we were also to stipulate that all of regularities $\{z_i = \psi_i(X)\}$ are proximal in $\langle y,Z,X\rangle$. However, we do **not** make this last assumption. Rather, our disconnection premise allows that some variables in Z may well be proximal sources in $\langle y,Z,X\rangle$ of other Z-variables, so that X's determination of some z_i in Z is mediated by Z-not-z_i. Prima facie, this is the very sort of proximality failure that invalidates FT-1 (p. 2.22). But applied to our simplified case, the argument for Theorem 15 observes that starting with $y = \phi(z_1,\ldots,z_m)$ proximal in $W =_{def} \langle y,Z,X\rangle$, X is the proximal source in W of at least one Z-variable, say z_1, under regularity $z_1 = \psi_1(X)$, so that $y = \phi(\psi_1(X),z_2,\ldots,z_m)$ is proximal in W-not-z_1. And all of regularities $z_2 = \psi_2(X),\ldots, z_m = \psi_m(X)$ are not only in W but are also in W-not-z_1 regardless of whether any are mediated by z_1, and at least one, say the first, must be proximal in W-not-z_1. So $z = \phi(\psi_1(X),\psi_2(X),z_3,\ldots,z_m)$ is proximal in W-not-$\langle z_1,z_2\rangle$ while $z_3 = \psi_3(X),\ldots, z_m = \psi_m(X)$ are all in W-not-$\langle z_1,z_2\rangle$ with at least one proximal therein. Iteration of this reduction eventually gives $y = \phi(\psi_1(X),\ldots,\psi_m(X))$ as proximal within W-not-$Z = \langle y,X\rangle$. Need for Z to disconnect X from y here is to yield that if $y = \phi(Z)$ is strictly causal, it is also proximal in $\langle y,Z,X\rangle$; after that in the reduction, the proximalities take care of themselves. As for our simplifying assumption that X is disjoint from and strictly determines all of Z, it is not hard to see that this plays no role in the argument except to suppress irrelevant distractions.

Composable sequences of macrocausal determination.

Theorem 15 may well be viewed as The Fundamental Principle of Causal Macrostructure--or indeed, of multivariate causal analysis in general. Given that causal structure is of dubious significance unless accompanied by causal composability, the main task of macrostructural theory is to work out the conditions under which the composability described by Theorem 15 can be iterated throughout complexes of Tuples in molar counterparts of microcausal paths.

Consider a sequence $X_1 \Rightarrow X_2 \Rightarrow \ldots \Rightarrow X_m \Rightarrow X_{m+1}$ of s-determinations. Under what circumstances is the quasi-causal regularity under which X_1 s-determines X_{m+1}

simply the serial composition of the quasi-causal regularities under which each $\underset{\wedge}{X}_i$ s-determines $\underset{\wedge}{X}_{i+1}$ in this sequence? The essential condition for this is given by

> *Definition* 2.18. A sequence (*not* a conjoined Tuple) $\underset{\wedge}{X}_1;\underset{\wedge}{X}_2;\ldots;\underset{\wedge}{X}_m;\underset{\wedge}{X}_{m+1}$ of Tuples is a (*length-m*) *composable determination series*, or *cd-series* for short, iff (*a*) $m \geq 2$, (*b*) for each $i = 1,\ldots,m$, $\underset{\wedge}{X}_i$ s-determines $\underset{\wedge}{X}_{i+1}$, and ($c_1$) $\underset{\wedge}{X}_2$ disconnects $\underset{\wedge}{X}_1$ from $\underset{\wedge}{X}_3$ if $m = 2$, or (c_2), if $m > 2$, there is some $h = 2,\ldots,m$ such that $\underset{\wedge}{X}_h$ disconnects $\underset{\wedge}{X}_{h-1}$ from $\underset{\wedge}{X}_{h+1}$ while $\underset{\wedge}{X}_1,\ldots,\underset{\wedge}{X}_{h-1},\underset{\wedge}{X}_{h+1},\ldots,\underset{\wedge}{X}_{m+1}$ is a length-(m-1) cd-series. (Note that this definition is a recursion on sequence length with $m = 2$ as base.)

We shall also write $\underset{\wedge}{X}_1 \dot{\Rightarrow} \ldots \dot{\Rightarrow} \underset{\wedge}{X}_{m+1}$ for sequences of tuples for which it is given that each $\underset{\wedge}{X}_i$ s-determines $\underset{\wedge}{X}_{i+1}$.

> *Theorem* 16. Let $\underset{\wedge}{X}_1 \dot{\Rightarrow} \underset{\wedge}{X}_2 \dot{\Rightarrow} \ldots \dot{\Rightarrow} \underset{\wedge}{X}_m \dot{\Rightarrow} \underset{\wedge}{X}_{m+1}$ be a sequence of s-determinations in which, for each $i = 1,\ldots,m$, $\underset{\wedge}{X}_i$ determines $\underset{\wedge}{X}_{i+1}$ under quasi-causal regularity $\underset{\wedge}{X}_{i+1} = \phi_i(\underset{\wedge}{X}_i)$. If this sequence is moreover a cd-series, then $\underset{\wedge}{X}_1$ s-determines $\underset{\wedge}{X}_{m+1}$ under the quasi-causal regularity $\underset{\wedge}{X}_{m+1} = \phi^*(\underset{\wedge}{X}_1)$ whose transducer is $\phi^* = \phi_m\phi_{m-1}\ldots\phi_2\phi_1$.

Proof, by induction on m. Given that $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is a cd-series, the induction's base is immediate from Theorem 15 when $m = 2$. More generally, for $m > 2$, let $\underset{\wedge}{X}_h$ be a tuple that disconnects $\underset{\wedge}{X}_{h-1}$ from $\underset{\wedge}{X}_{h+1}$ while $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{h-1};\underset{\wedge}{X}_{h+1};\ldots;\underset{\wedge}{X}_{m+1}$ is a length-(m-1) cd-series. (Existence of this $\underset{\wedge}{X}_h$ is stipulated by Def. 2.18.) Then by the induction hypothesis, the quasi-causal regularity under which $\underset{\wedge}{X}_{m+1}$ is s-determined by $\underset{\wedge}{X}_1$ has transducer $\phi^* = \phi_m\ldots\phi_{h+1}\phi'\phi_{h-2}\ldots\phi_1$ where ϕ' is the transducer of the quasi-causal regularity $\underset{\wedge}{X}_{h+1} = \phi'(\underset{\wedge}{X}_{h-1})$ under which $\underset{\wedge}{X}_{h-1}$ s-determines $\underset{\wedge}{X}_{h+1}$. But since $\underset{\wedge}{X}_h$ disconnects $\underset{\wedge}{X}_{h-1}$ from $\underset{\wedge}{X}_{h+1}$, we have from Theorem 15 that $\phi' = \phi_h\phi_{h-1}$. So substitution of $\phi_h\phi_{h-1}$ for ϕ' in the induction-hypothesis composition of ϕ^* yields $\phi^* = \phi_m\ldots\phi_{m+1}\phi_h\phi_{h-1}\ldots\phi_1$. □

For reasons overbriefly sketched earlier (p. 2.28), the converse of Theorem 16 is also essentially true--i.e., if $\underset{\wedge}{X}_1 \dot{\Rightarrow} \ldots \dot{\Rightarrow} \underset{\wedge}{X}_{m+1}$, in order for the quasi-causal regularity under which $\underset{\wedge}{X}_1$ determines $\underset{\wedge}{X}_{m+1}$ to have as its transducer the serial composition of the single-step quasi-causal transducers in this sequence, it is not merely sufficient but for all practical purposes necessary that this s-determination sequence be a cd-series. Whether there are any theoretically significant ways in which violations of this virtual necessity can arise, I do not know.

Theorem 16 makes evident that models of macrocausal structure want their distinguished sequences of causal determination to be composable whenever possible. Indeed, the most salient task for the theory of causal macrostructure is to identify analytically well-behaved structural conditions that suffice for a given s-determination sequence to be a cd-series. Particularly wanted are principles under which the sequence's holistic (global) status as a cd-series derives from the local properties of its proper subsequences. Intuitively, for example, it seems as though composability should follow if each mediating stage $\underset{\wedge}{X}_i$ in $\underset{\wedge}{X}_1 \dot{\Rightarrow} \ldots \dot{\Rightarrow} \underset{\wedge}{X}_{m+1}$ disconnects $\underset{\wedge}{X}_{i-1}$ from $\underset{\wedge}{X}_{i+1}$. Yet that is not generally so, as illustrated by $\underset{\wedge}{W} = \langle \underset{\wedge}{w}_1, \underset{\wedge}{w}_2 \rangle$, $\underset{\wedge}{X} = \langle \underset{\wedge}{x} \rangle$, $\underset{\wedge}{Y} = \langle \underset{\wedge}{y} \rangle$, $\underset{\wedge}{Z} = \langle \underset{\wedge}{z}_1, \underset{\wedge}{z}_2 \rangle$ when the path digraph for $\langle \underset{\wedge}{W}, \underset{\wedge}{X}, \underset{\wedge}{Y}, \underset{\wedge}{Z} \rangle$ is

$$\underset{\wedge}{w}_1 \rightarrow \underset{\wedge}{z}_1 \rightarrow \underset{\wedge}{x} \rightarrow \underset{\wedge}{y} \rightarrow \underset{\wedge}{w}_2 \rightarrow \underset{\wedge}{z}_2 \ .$$

Here $\underset{\wedge}{W} \dot{\Rightarrow} \underset{\wedge}{X} \dot{\Rightarrow} \underset{\wedge}{Y} \dot{\Rightarrow} \underset{\wedge}{Z}$ while $\underset{\wedge}{X}$ disconnects $\underset{\wedge}{W}$ from $\underset{\wedge}{Y}$ and $\underset{\wedge}{Y}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Z}$; yet neither $\underset{\wedge}{X}$ nor $\underset{\wedge}{Y}$ disconnects $\underset{\wedge}{W}$ from $\underset{\wedge}{Z}$, so $\underset{\wedge}{W};\underset{\wedge}{X};\underset{\wedge}{Y};\underset{\wedge}{Z}$ is not a cd-series.

A more successful intuition is that composable determination is importantly related to the local-disconnection condition described by

<u>Definition</u> 2.19. A sequence $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ ($\underline{m} \geq 2$) of tuples is a <u>standard cd-series</u> iff (<u>a</u>) $\underset{\wedge}{X}_i \dot{\Rightarrow} \underset{\wedge}{X}_{i+1}$ for all $\underline{i} = 1,\ldots,\underline{m}$, and (<u>b</u>) for all $\underset{\wedge}{X}_i$, $\underset{\wedge}{X}_j$, and $\underset{\wedge}{X}_k$ in this sequence with $1 \leq \underline{i} < \underline{j} < \underline{k} \leq \underline{m}+1$, $\underset{\wedge}{X}_j$ disconnects $\underset{\wedge}{X}_i$ from $\underset{\wedge}{X}_k$. (<u>Corollary</u>. If $\underset{\wedge}{X}_1;\ldots\underset{\wedge}{X}_{m+1}$ is a standard cd-series, every subsequence thereof formed by deleting $\underline{m}$-2 or fewer of its stages is also a standard cd-series.)

It is easily seen by induction on sequence length that all standard cd-series are also cd-series. Unhappily for simplicity, however, the converse fails for $\underline{m} \geq 3$, as demonstrated by $\underset{\wedge}{W} = \langle \underset{\wedge}{w} \rangle$, $\underset{\wedge}{X} = \langle \underset{\wedge}{x}_1, \underset{\wedge}{x}_2 \rangle$, $\underset{\wedge}{Y} = \langle \underset{\wedge}{y} \rangle$, $\underset{\wedge}{Z} = \langle \underset{\wedge}{z} \rangle$ when the path digraph for $\langle \underset{\wedge}{W}, \underset{\wedge}{X}, \underset{\wedge}{Y}, \underset{\wedge}{Z} \rangle$ is

$$\underset{\wedge}{w} \rightarrow \underset{\wedge}{x}_1 \rightarrow \underset{\wedge}{y} \rightarrow \underset{\wedge}{x}_2 \rightarrow \underset{\wedge}{z} \; .$$

Clearly $\underset{\wedge}{W} \dot{\Rightarrow} \underset{\wedge}{X} \dot{\Rightarrow} \underset{\wedge}{Y} \dot{\Rightarrow} \underset{\wedge}{Z}$, while $\underset{\wedge}{W};\underset{\wedge}{Y};\underset{\wedge}{Z}$ is a cd-series and $\underset{\wedge}{X}$ disconnects $\underset{\wedge}{W}$ from $\underset{\wedge}{Y}$. So $\underset{\wedge}{W};\underset{\wedge}{X};\underset{\wedge}{Y};\underset{\wedge}{Z}$ is a cd-series, and indeed its composability can easily be confirmed; yet it is not a standard cd-series, for $\underset{\wedge}{Y}$ does not disconnect $\underset{\wedge}{X}$ from $\underset{\wedge}{Z}$. Even so, standard cd-series comprise the broadest category of composable causal sequences that is analytically perspicuous, and, as will be seen, include as special cases the composabilities that are represented in path digraphs.

Standard composability can be characterized in several ways. One useful variation, an immediate consequence of Def. 2.19 by Theorem 13-1, is

> <u>Theorem</u> 17. Let $\underset{\wedge}{X}_1 \dot{\Rightarrow} \ldots \dot{\Rightarrow} \underset{\wedge}{X}_{m+1}$ be an s-determination sequence and, for each $\underline{i} = 1,\ldots,\underline{m}+1$, stipulate
>
> $$\underset{\wedge}{X}_i^a =_{def} \langle \underset{\wedge}{X}_1,\ldots,\underset{\wedge}{X}_{i-1},\underset{\wedge}{X}_i \rangle \;, \qquad \underset{\wedge}{X}_i^c =_{def} \langle \underset{\wedge}{X}_i,\underset{\wedge}{X}_{i+1},\ldots,\underset{\wedge}{X}_{m+1} \rangle \; .$$
>
> (The superscripts in "$\underset{\wedge}{X}_i^a$" and "$\underset{\wedge}{X}_i^c$" are heuristic for "antecedent" and "consequent," respectively.) Then $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is a standard cd-series just in case, for each $\underline{i} = 2,\ldots,\underline{m}$, $\underset{\wedge}{X}_i$ disconnects $\underset{\wedge}{X}_i^a$ from $\underset{\wedge}{X}_i^c$ (equivalently, $\underset{\wedge}{X}_i^a$-not-$\underset{\wedge}{X}_i$ from $\underset{\wedge}{X}_i^c$-not-$\underset{\wedge}{X}_i$).

Henceforth we shall use "$\underset{\wedge}{X}_i^a$" and "$\underset{\wedge}{X}_i^c$" specifically as just defined, though concern for $\underset{\wedge}{X}_i^a$ will be fleeting. Note that for considering whether $\underset{\wedge}{X}_i$ disconnects $\underset{\wedge}{X}_i^a$ from $\underset{\wedge}{X}_i^c$, the tuple $\langle \underset{\wedge}{X}_i^a, \underset{\wedge}{X}_i, \underset{\wedge}{X}_i^c \rangle$ whose path structure ajudicates this (cf. Theorem 14) is just the sequence's union $\langle \underset{\wedge}{X}_1,\ldots,\underset{\wedge}{X}_{m+1} \rangle$, the same for all $\underline{i}$.

Theorem 17 is a technical convenience, but it does not much illuminate the nature of standard composability. We now observe, more deeply, that this derives from the structural properties described by

<u>Definition</u> 2.20. A sequence $\underset{\wedge}{X}_1;\ldots\underset{\wedge}{X}_{m+1}$ of tuples is <u>(repetitionwise) convex</u> iff, for all $\underset{\wedge}{X}_i$, $\underset{\wedge}{X}_j$, and $\underset{\wedge}{X}_k$ therein with $\underline{i} < \underline{j} < \underline{k}$, every variable common to $\underset{\wedge}{X}_i$ and $\underset{\wedge}{X}_k$ is also in $\underset{\wedge}{X}_j$. (Equivalently, $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is convex iff, for each $\underline{i}$ = 2,...,$\underline{m}$, $\underset{\wedge}{X}_i^a$-not-$\underset{\wedge}{X}_i$ and $\underset{\wedge}{X}_i^c$-not-$\underset{\wedge}{X}_i$ are disjoint.)

<u>Definition</u> 2.21. A sequence $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ of tuples is <u>compact within</u> $\underset{\wedge}{Z}$ (equivalently, $\underset{\wedge}{Z}$-<u>wise compact</u>) iff $\underset{\wedge}{Z}$ is a tuple that includes all variables in $\langle\underset{\wedge}{X}_1,\ldots,\underset{\wedge}{X}_{m+1}\rangle$ and, for all $\underline{i}$ = 1,...,$\underline{m}$ and each variable $\underset{\wedge}{x}$ in $\underset{\wedge}{X}_i^c$-not-$\underset{\wedge}{X}_i$, $\underset{\wedge}{X}_i^c$ includes all direct sources of $\underset{\wedge}{x}$ within $\underset{\wedge}{Z}$. (Note that this definition holds for case $\underline{m}$ = 1 as well as normal case $\underline{m} \geq 2$, and that $\underset{\wedge}{X}_i^c$-not-$\underset{\wedge}{X}_i \doteq \underset{\wedge}{X}_{i+1}^c$-not-$\underset{\wedge}{X}_i$.) Sequence $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is $\underset{\wedge}{Z}$-wise <u>strongly compact</u> iff, for all $\underline{i}$ = 2,...,$\underline{m}$, $\underset{\wedge}{X}_i;\underset{\wedge}{X}_{i+1}$ is $\underset{\wedge}{Z}$-wise compact. Sequence $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is <u>(intrinsically) compact</u> iff it is compact within $\langle\underset{\wedge}{X}_1,\ldots,\underset{\wedge}{X}_{m+1}\rangle$.

Any sequence that is $\underset{\wedge}{Z}$-wise strongly compact is also $\underset{\wedge}{Z}$-wise compact; however, what strong compactness adds to compactness <u>simpliciter</u> will not concern us for some time. More immediately relevant is that if sequence $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is $\underset{\wedge}{Z}$-wise compact, and $\underset{\wedge}{Z}_0$ contains only variables in $\underset{\wedge}{Z}$-not-$\langle\underset{\wedge}{X}_1,\ldots,\underset{\wedge}{X}_{m+1}\rangle$, no direct source within $\underset{\wedge}{Z}$ of any variable $\underset{\wedge}{x}$ in $\underset{\wedge}{X}_i^c$-not-$\underset{\wedge}{X}_i$ for any $\underline{i}$ = 2,...,$\underline{m}$ is in $\underset{\wedge}{Z}_0$; hence $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is also compact within $\underset{\wedge}{Z}$-not-$\underset{\wedge}{Z}_0$ insomuch as the proximal source within $\underset{\wedge}{Z}$ of each $\underset{\wedge}{x}$ in each $\underset{\wedge}{X}_i^c$-not-$\underset{\wedge}{X}_i$ is then also a strictly complete and moreover proximal source of $\underset{\wedge}{x}$ within $\underset{\wedge}{Z}$-not-$\underset{\wedge}{Z}_0$. So every sequence of tuples that is compact in some $\underset{\wedge}{Z}$ is intrinsically compact as well. (Conversely, however, a sequence that is $\underset{\wedge}{Z}$-wise compact may not be compact within a proper supertuple $\underset{\wedge}{Z}'$ of $\underset{\wedge}{Z}$, since $\underset{\wedge}{Z}'$-not-$\underset{\wedge}{Z}$ may contain mediators of the direct-source connections in $\underset{\wedge}{Z}$.) Note also that if $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is compact, each variable in $\underset{\wedge}{X}_i^c$-not-$\underset{\wedge}{X}_i$ ($\underline{i}$ = 1,...,$\underline{m}$) is interior to $\underset{\wedge}{X}_i^c$, so $\underset{\wedge}{X}_i$ and $\underset{\wedge}{X}_i^c$ have the same exterior and $\underset{\wedge}{X}_i \doteq\!> \underset{\wedge}{X}_i^c \doteq\!> \underset{\wedge}{X}_{i+1}$. Hence any compact sequence of tuples is an s-determination sequence.

<u>Theorem</u> 18. Any sequence $\mathbf{X}_1;\ldots;\mathbf{X}_{m+1}$ of tuples is a standard cd-series just in case it is both repetitionwise convex and intrinsically compact.

<u>Corollary</u>. If sequence $\mathbf{X}_1;\ldots;\mathbf{X}_{m+1}$ is convex and compact within any tuple $\mathbf{Z}$, $\mathbf{X}_1;\ldots;\mathbf{X}_{m+1}$ is a standard cd-series.

<u>Proof</u>. First we show necessity. Clearly the sequence must be convex if it is a cd-series; for if any $\mathbf{X}_j$ therein lacks some variable common to $\mathbf{X}_i$ and $\mathbf{X}_k$ ($\underline{i} < \underline{j} < \underline{k}$), $\mathbf{X}_j$ would not disconnect $\mathbf{X}_i$ from $\mathbf{X}_k$ (cf. Theorem 14). And if, in violation of compactness, some variable $\mathbf{x}$ in $\mathbf{X}_i^c$-not-$\mathbf{X}_i$ were to have a direct source $\mathbf{x}'$ within $\langle \mathbf{X}_1,\ldots,\mathbf{X}_{m+1}\rangle$ that is not in $\mathbf{X}_i^c$ (which is possible only if $2 \leq \underline{i} \leq \underline{m}$), $\mathbf{x}'$ would be a variable in $\mathbf{X}_i^a$-not-$\mathbf{X}_i$ that is a direct source within $\langle \mathbf{X}_i^a,\mathbf{X}_i,\mathbf{X}_i^c\rangle (= \langle \mathbf{X}_1,\ldots,\mathbf{X}_{m+1}\rangle)$ of a variable in $\mathbf{X}_i^c$-not-$\mathbf{X}_i$, namely $\mathbf{x}$, so $\mathbf{X}_i$ would not disconnect $\mathbf{X}_i^a$ from $\mathbf{X}_i^c$ and $\mathbf{X}_1;\ldots;\mathbf{X}_{m+1}$ would hence not be a standard cd-series (cf. Theorem 17). Conversely, suppose that sequence $\mathbf{X}_1;\ldots;\mathbf{X}_{m+1}$ is both convex and compact. We have already observed that compactness makes this an s-determination sequence. So by Theorems 17 & 14, it suffices to show, for all $\underline{i} = 2,\ldots,\underline{m}$ both that each variable common to $\mathbf{X}_i^a$ and $\mathbf{X}_i^c$ is also in $\mathbf{X}_i$--which follows immediately from the sequence's stipulated convexity--and that each path $\mathbf{X}_{jk}$ within $\langle \mathbf{X}_i^a,\mathbf{X}_i,\mathbf{X}_i^c\rangle$ from a variable $\mathbf{x}_j$ in $\mathbf{X}_i^a$-not-$\mathbf{X}_i$ to a variable $\mathbf{x}_k$ in $\mathbf{X}_i^c$-not-$\mathbf{X}_i$ passes through $\mathbf{X}_i$. Suppose to the contrary, for disproof, that some such path $\mathbf{X}_{jk}$ were not to pass through $\mathbf{X}_i$. Then $\mathbf{X}_{jk}$ would have to contain at least one variable $\mathbf{x}_j'$ in $\mathbf{X}_i^a$-not-$\mathbf{X}_i$ immediately followed by a variable $\mathbf{x}_k'$ in $\mathbf{X}_i^c$-not-$\mathbf{X}_i$, which is to say that $\mathbf{x}_j'$ is a direct source of $\mathbf{x}_k'$ within $\langle \mathbf{X}_1,\ldots,\mathbf{X}_{m+1}\rangle$. Unless this $\mathbf{x}_j'$ were to be in $\mathbf{X}_i^c$, $\langle \mathbf{x}_j',\mathbf{x}_k'\rangle$ would violate the stipulation that $\mathbf{X}_1;\ldots;\mathbf{X}_{m+1}$ is compact. But since $\mathbf{x}_j'$ is in $\mathbf{X}_i^a$-not-$\mathbf{X}_i$ it cannot be in $\mathbf{X}_i^c$ without violating convexity. So convexity and compactness together suffice for a tuple sequence to be a standard cd-series. The corollary is immediate from our previous observation that $\mathbf{Z}$-wise compactness for any $\mathbf{Z}$ entails intrinsic compactness. □

Given a background tuple $\mathbf{X}$ within which we know (or hypothesize) the micro-causal path structure, Theorem 18 Corollary tells us how to construct standard cd-series of $\mathbf{X}$'s subtuples. To make this perspicuous, let us start with a close look at the generic structure of any s-determination sequence $\underset{\wedge}{X}_1 \doteq\!> \ldots \doteq\!> \underset{\wedge}{X}_{m+1}$. For each $\underline{i} = 1,\ldots,\underline{m}$, write

$$\underset{\wedge}{X}^{o}_{i+1} =_{def} \underset{\wedge}{X}_{i+1}\text{-not-}\underset{\wedge}{X}_{i}\ , \qquad \underset{\wedge}{X}^{+}_{i} =_{def} \underset{\wedge}{X}_{i}\text{-not-}\underset{\wedge}{X}_{i+1}\ .$$

Superscripts "o" and "+" here are heuristic for "omission" and "addition," respectively. Our interest in $\underset{\wedge}{X}^{+}_{i}$ will soon transfer to another subtuple $\underset{\wedge}{X}'_{i}$ of $\underset{\wedge}{X}_{i}$ more inclusive than $\underset{\wedge}{X}^{+}_{i}$, but $\underset{\wedge}{X}^{o}_{i}$ will be important for the remainder of this section. Moreover, since we shall repeatedly refer to the aggregate of all omissions $\underset{\wedge}{X}^{o}_{j}$ ($\underline{j} = \underline{i}{+}1,\ldots,\underline{m}{+}1$) from stages of the series following $\underset{\wedge}{X}_{i}$, it will also be convenient to write

$$\underset{\wedge}{X}^{oo}_{i+1} =_{def} \langle \underset{\wedge}{X}^{o}_{i+1}, \underset{\wedge}{X}^{o}_{i+2}, \ldots, \underset{\wedge}{X}^{o}_{m+1} \rangle .$$

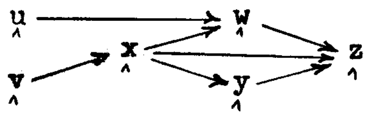
Viewing sequence $\underset{\wedge}{X}_1 \doteq\!> \ldots \doteq\!> \underset{\wedge}{X}_{m+1}$ from right to left as a precession of quasi-causes, we can think of $\underset{\wedge}{X}^{+}_{i}$ as comprising whatever variables not already in $\underset{\wedge}{X}_{i+1}$ are picked up by $\underset{\wedge}{X}_{i}$, while $\underset{\wedge}{X}^{o}_{i}$ comprises the variables in $\underset{\wedge}{X}_{i}$ that are not retained in $\underset{\wedge}{X}_{i-1}$. ($\underset{\wedge}{X}^{oo}_{i}$ accumulates all omissions back through stage $\underset{\wedge}{X}_{i}$.) Thus $\underset{\wedge}{X}_{i} \doteq \langle \underset{\wedge}{X}^{+}_{i}, \underset{\wedge}{X}_{i+1} \rangle\text{-not-}\underset{\wedge}{X}^{o}_{i+1} \doteq \langle \underset{\wedge}{X}^{+}_{i}, \underset{\wedge}{X}_{i+1}\text{-not-}\underset{\wedge}{X}^{o}_{i+1} \rangle$. Either or both of $\underset{\wedge}{X}^{+}_{i}$ and $\underset{\wedge}{X}^{o}_{i}$ can be null; however, in the cases that interest us, any sequence stage $\underset{\wedge}{X}_{i}$ for which $\underset{\wedge}{X}^{o}_{i}$ is null is a triviality that can be removed by deleting $\underset{\wedge}{X}_{i}$ from the sequence. Hence for simplicity and without essential loss of generality we presume that each $\underset{\wedge}{X}^{o}_{i}$ is non-null. In contrast, assuming $\underset{\wedge}{X}^{+}_{i}$ to be non-null is appropriate only if we impose the additional constraint that $\underset{\wedge}{X}_{i+1}$ has null interior. Given that each $\underset{\wedge}{X}_{i}$ s-determines $\underset{\wedge}{X}_{i+1}$, the right-to-left precessional view of this sequence takes each $\underset{\wedge}{X}_{i}$ to be constructed--conceptually, not causally--from $\underset{\wedge}{X}_{i+1}$ by replacing $\underset{\wedge}{X}_{i+1}$'s subtuple $\underset{\wedge}{X}^{o}_{i+1}$ by some inclusively complete source $\underset{\wedge}{X}'_{i}$ of $\underset{\wedge}{X}^{o}_{i+1}$, i.e. $\underset{\wedge}{X}'_{i} \doteq\!> \underset{\wedge}{X}^{o}_{i+1}$ with $\underset{\wedge}{X}'_{i}$ disjoint from $\underset{\wedge}{X}^{o}_{i+1}$, while $\underset{\wedge}{X}^{+}_{i}$ then comprises whatever variables in $\underset{\wedge}{X}'_{i}$ are not already in $\underset{\wedge}{X}_{i}$. Subtuples $\underset{\wedge}{X}^{+}_{i}$ and $\underset{\wedge}{X}^{o}_{i}$ of $\underset{\wedge}{X}_{i}$ may or may not be disjoint; in any case, there is no conceptual tie between them. That is, if sequence $\underset{\wedge}{X}_1 \doteq\!> \ldots \doteq\!> \underset{\wedge}{X}_{m+1}$ is constructed by a recursive precession in which, for each $\underline{i} = \underline{m}{+}1, \underline{m}, \underline{m}{-}1, \ldots, 2$, we first identify $\underset{\wedge}{X}_{i};\ldots;\underset{\wedge}{X}_{m+1}$ and then choose which tuple $\underset{\wedge}{X}_{i-1}$ is to s-determine $\underset{\wedge}{X}_{i}$, we are free in principle to put any $\underset{\wedge}{X}_{i}$-variables we wish into omission tuple $\underset{\wedge}{X}_i^o$ so long as they are in $\underline{I}(\underset{\wedge}{X})$, regardless of whether they are new additions in $\underset{\wedge}{X}_i^+$ or carryovers from $\underset{\wedge}{X}_{i+1}$-not-$\underset{\wedge}{X}_{i+1}^o$.

From this precessional perspective on $\underset{\wedge}{X}_1 \doteq> \ldots \doteq> \underset{\wedge}{X}_{m+1}$, it is evident that each $\underset{\wedge}{X}_i$ comprises, in addition to $\underset{\wedge}{X}_i^+$, all variables in $\underset{\wedge}{X}_{i+1}^c$ $(= \langle \underset{\wedge}{X}_{i+1}, \ldots, \underset{\wedge}{X}_{m+1} \rangle)$ less some or all of the ones omitted in $\underset{\wedge}{X}_{i+1}^{oo}$ $(= \langle \underset{\wedge}{X}_{i+1}^o, \ldots, \underset{\wedge}{X}_{m+1}^o \rangle)$ That is, all variables in $\underset{\wedge}{X}_i^c$-not-$\underset{\wedge}{X}_i$ ($\doteq \underset{\wedge}{X}_{i+1}^c$-not-$\underset{\wedge}{X}_i$) are in $\underset{\wedge}{X}_{i+1}^{oo}$. A variable in $\underset{\wedge}{X}_k^o$ $(\underline{k} > \underline{i})$ can still be in $\underset{\wedge}{X}_i$ if it reappears in $\underset{\wedge}{X}_j^+$ for some $\underline{i} \le \underline{j} < \underline{k}-1$. But if $\underset{\wedge}{X}_1; \ldots; \underset{\wedge}{X}_{m+1}$ is repetitionwise convex and $\underset{\wedge}{X}_i$ hence disjoint from $\underset{\wedge}{X}_{i+1}^{oo}$, then $\underset{\wedge}{X}_i \doteq \underset{\wedge}{X}_i^c$-not-$\underset{\wedge}{X}_{i+1}^{oo}$ and $\underset{\wedge}{X}_i^c$-not-$\underset{\wedge}{X}_i \doteq \underset{\wedge}{X}_{i+1}^{oo}$.

Suppose, now, that starting with a given subtuple $\underset{\wedge}{X}_{m+1}$ of $\underset{\wedge}{X}$, we wish to specify by recursive precession an s-determination sequence $\underset{\wedge}{X}_1 \doteq> \ldots \doteq> \underset{\wedge}{X}_{m+1}$ that is moreover a standard cd-series. In light of Theorem 18 Corollary this is straightforward in principle: For each $\underline{i} = \underline{m}, \underline{m}-1, \ldots$ we select any subtuple $\underset{\wedge}{X}_i^o$ that we choose to eliminate at this precession stage in favor of its sources in $\underset{\wedge}{X}$, let $\underset{\wedge}{X}_{i-1}^!$ include every variable in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_i^c$ that is a direct source within $\underset{\wedge}{X}$ of some variable in $\underset{\wedge}{X}_i^o$, and also put into $\underset{\wedge}{X}_{i-1}^!$

any other $\underset{\wedge}{X}$-variables we may want there subject to the proviso that $\underset{\wedge}{X}_i^!$ is not to include any variable in $\underset{\wedge}{X}_i^{oo}$. Then taking $\underset{\wedge}{X}_{i-1} \doteq \langle \underset{\wedge}{X}_{i-1}^!, \underset{\wedge}{X}_i$-not-$\underset{\wedge}{X}_i^o \rangle$ continues the s-determination precession in $\underset{\wedge}{X}$. Including in each $\underset{\wedge}{X}_{i-1}$ all direct sources of $\underset{\wedge}{X}_i^o$ within $\underset{\wedge}{X}$ that are not already in $\underset{\wedge}{X}_i^c$ makes this sequence compact within $\underset{\wedge}{X}$ and hence also intrinsically compact, while compliance with the proviso insures that the sequence is moreover repetitionwise convex and hence, by Theorem 18 Corollary, a cd-series.

There is, however, one important limitation on this construction. At each precession stage, given $\underset{\wedge}{X}_i \doteq> \ldots \doteq> \underset{\wedge}{X}_{m+1}$ convex and $\underset{\wedge}{X}$-wise compact, and with some variable in $\underset{\wedge}{X}_i$ still interior to $\underset{\wedge}{X}$, we can always choose $\underset{\wedge}{X}_{i-1}$ nontrivially (i.e. $\underset{\wedge}{X}_i^o$ not null) to keep the extended sequence $\underset{\wedge}{X}_{i-1} \doteq> \underset{\wedge}{X}_i \doteq> \ldots \doteq> \underset{\wedge}{X}_{m+1}$ compact within $\underset{\wedge}{X}$. It is not, however, always possible for choice of $\underset{\wedge}{X}_{i-1}$ to preserve convexity while continuing $\underset{\wedge}{X}$-wise compactness. For example, suppose that the path digraph for $\underset{\wedge}{X} = \langle \underset{\wedge}{u}, \underset{\wedge}{v}, \underset{\wedge}{w}, \underset{\wedge}{x}, \underset{\wedge}{y}, \underset{\wedge}{z} \rangle$ is



If we take $\underset{\wedge}{X}_6 = \langle \underset{\wedge}{z} \rangle$, $\underset{\wedge}{X}_5 = \langle \underset{\wedge}{w},\underset{\wedge}{x},\underset{\wedge}{y} \rangle$, and $\underset{\wedge}{X}_4 = \langle \underset{\wedge}{w},\underset{\wedge}{v},\underset{\wedge}{y} \rangle$ with corresponding omissions $\underset{\wedge}{X}_6^o = \langle \underset{\wedge}{z} \rangle$ and $\underset{\wedge}{X}_5^o = \langle \underset{\wedge}{x} \rangle$, sequence $\underset{\wedge}{X}_4 \doteq\!> \underset{\wedge}{X}_5 \doteq\!> \underset{\wedge}{X}_6$ is both convex and $\underset{\wedge}{X}$-wise compact. If we next choose $\underset{\wedge}{X}_4^o = \langle \underset{\wedge}{y} \rangle$, continuing the precession by $\langle \underset{\wedge}{w},\underset{\wedge}{v},\underset{\wedge}{x} \rangle \doteq\!> \underset{\wedge}{X}_4$ would violate convexity; however, since $\underset{\wedge}{y}$'s proximal source within $\underset{\wedge}{X}$ is already in $\underset{\wedge}{X}_4^c$, we can instead take $\underset{\wedge}{X}_3 = \langle \underset{\wedge}{w},\underset{\wedge}{v} \rangle$ and have $\underset{\wedge}{X}_3 \doteq\!> \underset{\wedge}{X}_4 \doteq\!> \underset{\wedge}{X}_5 \doteq\!> \underset{\wedge}{X}_6$ still convex and compact in $\underset{\wedge}{X}$. But the only $\underset{\wedge}{X}$-wise compact continuation of that, in turn, is $\underset{\wedge}{X}_3^o = \langle \underset{\wedge}{w} \rangle$ with $\underset{\wedge}{X}_2 = \langle \underset{\wedge}{u},\underset{\wedge}{x},\underset{\wedge}{v} \rangle$. (From there we finish with $\underset{\wedge}{X}_2^o = \langle \underset{\wedge}{x} \rangle$ and $\underset{\wedge}{X}_1 = \langle \underset{\wedge}{u},\underset{\wedge}{v} \rangle$, which is as far as the precession can be carried in $\underset{\wedge}{X}$.) Sequence $\underset{\wedge}{X}_1 \doteq\!> \underset{\wedge}{X}_2 \doteq\!> \underset{\wedge}{X}_3 \doteq\!> \underset{\wedge}{X}_4 \doteq\!> \underset{\wedge}{X}_5 \doteq\!> \underset{\wedge}{X}_6$ is still $\underset{\wedge}{X}$-wise compact, but it is not convex insomuch as $\underset{\wedge}{x}$ is in both $\underset{\wedge}{X}_2$ and $\underset{\wedge}{X}_5$ but not in $\underset{\wedge}{X}_3$ or $\underset{\wedge}{X}_4$. And neither is this sequence a cd-series, standard or otherwise, when extended backward from $\underset{\wedge}{X}_3$. Indeed, this example could easily be used instead of the ones based on Figs. 1 & 2 to illustrate non-composability.

One way to avoid convexity violations when constructing a standard cd-series is to specify a compact s-determination sequence $\underset{\wedge}{X}_1 \doteq\!> \ldots \doteq\!> \underset{\wedge}{X}_{m+1}$ in the fashion just described without concern for convexity, and afterward, for each $\underset{\wedge}{X}_i$ and $\underset{\wedge}{X}_k$, to add each variable common to $\underset{\wedge}{X}_i$ and $\underset{\wedge}{X}_k$ to every $\underset{\wedge}{X}_j$ between $\underset{\wedge}{X}_i$ and $\underset{\wedge}{X}_k$ that lacks it. Thus in the example just given, if $\underset{\wedge}{x}$ is added to the original $\underset{\wedge}{X}_3$ and $\underset{\wedge}{X}_4$ to convert these to $\underset{\wedge}{X}_3 = \langle \underset{\wedge}{w},\underset{\wedge}{v},\underset{\wedge}{x} \rangle$ and $\underset{\wedge}{X}_4 = \langle \underset{\wedge}{w},\underset{\wedge}{v},\underset{\wedge}{y},\underset{\wedge}{x} \rangle$, the modified $\underset{\wedge}{X}_1 \doteq\!> \ldots \doteq\!> \underset{\wedge}{X}_6$ is now a standard cd-series. However, this afterthought-convexification procedure does not identify standard cd-series by iterating their precessions.

Alternatively, if we want standard cd-series whose precessions can be continued systematically in counterpart to the causal composabilities implicit in microcausal path digraphs, we need some additional constraints on tuples $\underset{\wedge}{X}_i^o$ and $\underset{\wedge}{X}_{i-1}'$ at each stage of the sequence's precession. These constraints are defined in terms of the path structure within the background tuple $\underset{\wedge}{X}$ comprising all variables in the more local tuples whose causal connections are at issue, and are based on relativations to $\underset{\wedge}{X}$ of causal concepts previously defined in absolute terms (cf. previous comments on this, p. 2.47):

<u>Definition</u> 2.22. For any background tuple $\underset{\wedge}{X}$: (1) Variable $\underset{\wedge}{y}$ is an $\underset{\wedge}{X}$-<u>wise</u> (<u>causal</u>) <u>source</u> of variable $\underset{\wedge}{z}$ iff there is a causal path from $\underset{\wedge}{y}$ to $\underset{\wedge}{z}$ within $\underset{\wedge}{X}$. Informally, we will also say that $\underset{\wedge}{x}$ is an $\underset{\wedge}{X}$-wise source of tuple $\underset{\wedge}{Z}$ iff $\underset{\wedge}{x}$ is an $\underset{\wedge}{X}$-wise source of some variable in $\underset{\wedge}{Z}$. (2) Variable $\underset{\wedge}{y}$ is an $\underset{\wedge}{X}$-<u>wise</u> <u>direct</u> <u>source</u> of variable $\underset{\wedge}{z}$ iff $\underset{\wedge}{y}$ is a direct source of $\underset{\wedge}{z}$ within $\underset{\wedge}{X}$. (3) A tuple $\underset{\wedge}{Y}$ is $\underset{\wedge}{X}$-<u>wise</u> <u>c(ausally)</u> <u>independent</u> of tuple $\underset{\wedge}{Z}$ iff no variable in $\underset{\wedge}{Y}$ is either in $\underset{\wedge}{Z}$ or is an $\underset{\wedge}{X}$-wise source of any variable in $\underset{\wedge}{Z}$. (4) A sequence $\underset{\wedge}{Y}_1;\ldots;\underset{\wedge}{Y}_n$ of tuples is $\underset{\wedge}{X}$-<u>wise</u> <u>well</u>-<u>ordered</u> iff, for all $\underline{i},\underline{j} = 1,\ldots,\underline{n}$ with $\underline{i} < \underline{j}$, $\underset{\wedge}{Y}_i$ is $\underset{\wedge}{X}$-wise c-independent of $\underset{\wedge}{Y}_j$. (5) Tuple Y <u>t_X-precedes</u> tuple $\underset{\wedge}{Z}$ (i.e., $\underset{\wedge}{Y}$ t-precedes $\underset{\wedge}{Z}$ relative to the $\underset{\wedge}{X}$-wise causal-source relation) iff each variable in $\underset{\wedge}{Y}$ is an $\underset{\wedge}{X}$-wise source of some variable in $\underset{\wedge}{Z}$ (cf. Def. 2.11). (6) Tuple $\underset{\wedge}{Y}$ <u>t_X-determines</u> tuple $\underset{\wedge}{Z}$ (i.e., $\underset{\wedge}{X}$ t-determines $\underset{\wedge}{Y}$ relative to the $\underset{\wedge}{X}$-wise causal-source relation) iff $\underset{\wedge}{Y}$ s-determines $\underset{\wedge}{Z}$ and $\underset{\wedge}{Y}$-not-$\underset{\wedge}{Z}$ t_X-precedes $\underset{\wedge}{Z}$-not-$\underset{\wedge}{X}$ (cf. Def. 2.14 and Theorem 10).

Evidently, the $\underset{\wedge}{X}$-wise source and $t_{\underset{\wedge}{X}}$-precedence relations have the same partial-order character as their absolute counterparts. And $\underset{\wedge}{Y}$ t_X-determines $\underset{\wedge}{Z}$ only if $\underset{\wedge}{Y}$ (absolutely) t-determines $\underset{\wedge}{Z}$. Moreover, t_X-determination is transitive and classically anti-symmetric by the very same argument that establishes this for t-determination--we merely replace the (absolute) causal-source relation in the original proof by the $\underset{\wedge}{X}$-wise source relation.

To achieve convexity of molar determination sequences systematically, our first constraint is that each $\underset{\wedge}{X}_i$ in sequence $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ of $\underset{\wedge}{X}$-subtuples is not merely to s-determine $\underset{\wedge}{X}_{i+1}$ but to t_X-determine it. This is equivalent to requiring for each $\underline{i} = 1,\ldots,\underline{m}$ that there be a path within $\underset{\wedge}{X}$ from each variable in $\underset{\wedge}{X}_i^+$ to some variable in $\underset{\wedge}{X}_{i+1}^o$. And our second constraint is that the sequence $\underset{\wedge}{X}_2^o;\ldots;\underset{\wedge}{X}_{m+1}^o$ of omissions from our t_X-determination sequence is to be $\underset{\wedge}{X}$-wise well-ordered. This yields

<u>Theorem</u> 19. Let $X_1 \Rightarrow \ldots \Rightarrow X_{m+1}$ be a t_X-determination sequence of X-subtuples in which $X_2^o;\ldots;X_{m+1}^o$ is X-wise well-ordered. That is, each X_i ($\underline{i} = 1, \ldots, \underline{m}$) t_X-determines X_{i+1} and each X_i^o ($\underline{i} = 2, \ldots, \underline{m}$) is X-wise causally independent of X_{i+1}^{oo}. Then sequence $X_1;\ldots;X_{m+1}$ is repetitionwise convex; so if it is also X-wise compact, it is a standard cd-series that is moreover X-wise strongly compact (cf. Def. 2.21).

<u>Proof</u>. Let $X_1;\ldots;X_{m+1}$ be as stipulated, and conjecture that for some $\underline{i} = 2, \ldots, \underline{m}$ and $\underline{j} > \underline{i}$, some variable x common to X_{i-1} and X_j is not in X_i. Since all variables in $\langle X_{i+1}, \ldots, X_{m+1}\rangle$ but not in X_i are in X_{i+1}^{oo}, x must be in the latter. Now by hypothesis x is in X_{i-1}-not-X_i and so by t_X-determination has to be an X-wise source of some variable x' in X_i^o. But then some variable in X_{i+1}^{oo}, namely x, would be an X-wise source of some variable in X_i^o, namely x', contrary to presumption that X_i^o is X-wise c-independent of X_{i+1}^{oo}. So the conjecture is disproved, showing that for all $\underline{i} < \underline{j}$, any variable common to X_{i-1} and X_j must also be in X_i--which is to say that sequence $X_1 \Rightarrow \ldots \Rightarrow X_{m+1}$ is repetitionwise convex. With compactness also stipulated, it follows from Theorem 18 that the sequence is a standard cd-series. Moreover, for each $\underline{i} = 1, \ldots, \underline{m}$, all X-wise direct sources of each variable x in X_{i+1}^o must be in $\langle X_i, X_{i+1}\rangle$ ($= \langle X_i, X_{i+1}^o\rangle$), for by compactness, every X-wise direct source of x must be in X_i^c and cannot be in X_{i+2}^{oo} else X_{i+1}^o would not be X-wise c-independent of the latter. □

The special properties invoked in Theorem 19--t_X-determination, X-wise well-ordered omissions, and X-wise strong compactness--are essential for a sequence of X-subtuples to have representation in a macrocausal version of path structure. (For the significance of strong compactness, see Theorem 24 below.) But a little more is also needed, as realized in two stronger cases that are of special interest.

<u>Definition</u> 2.23. A tuple Z is X-<u>wise solid</u> iff all Z-variables are in X and Z includes all variables in every path within X from one Z-variable to another. (Equivalently, Z is X-wise solid iff X includes Z and, for all x_i, x_j, and x_k, if x_i is an X-wise source of x_j and x_j is an X-wise source of x_k, x_j is in Z if x_i and x_k are both in Z. Also, Z is X-wise solid iff every tuple disjoint from Z that t_X-precedes Z is X-wise causally independent of Z.)

<u>Definition</u> 2.24. (1) A t_X-determination sequence $\underset{\wedge}{X}_1 \Rightarrow \ldots \Rightarrow \underset{\wedge}{X}_{m+1}$ is $\underset{\wedge}{X}$-<u>wise chained</u> (equivalently, is an $\underset{\wedge}{X}$-wise chain) iff the sequence is $\underset{\wedge}{X}$-wise compact and, for each $\underline{i} = 2,\ldots,\underline{m}$, $\underset{\wedge}{X}^o_i$ t_X-precedes $\underset{\wedge}{X}^o_{i+1}$ with $\underset{\wedge}{X}^o_{i+1}$ $\underset{\wedge}{X}$-wise solid. (2) A t_X-determination sequence $\underset{\wedge}{X}_1 \Rightarrow \ldots \Rightarrow \underset{\wedge}{X}_{m+1}$ is $\underset{\wedge}{X}$-<u>wise solidly conservative</u> iff the sequence is $\underset{\wedge}{X}$-wise compact, $\underset{\wedge}{X}_{m+1}$-not-$\underset{\wedge}{X}^o_{m+1}$ t_X-precedes $\underset{\wedge}{X}^o_{m+1}$ (as holds in particular if $\underset{\wedge}{X}_{m+1} \doteq \underset{\wedge}{X}^o_{m+1}$), and for each $\underline{i} = 1,\ldots,\underline{m}$, $\underset{\wedge}{X}^{oo}_{i+1}$ is $\underset{\wedge}{X}$-wise solid.

Chained sequences are basic in causal macrostructure; for as will soon be noted, the t_X-precedence ordering $\underset{\wedge}{X}^o_2;\ldots;\underset{\wedge}{X}^o_{m+1}$ of omissions in an $\underset{\wedge}{X}$-wise chained t_X-determination sequence is the molar counterpart of a microstructural causal path. First, though, we observe

<u>Theorem</u> 20. Let $\underset{\wedge}{X}_1 \Rightarrow \ldots \Rightarrow \underset{\wedge}{X}_{m+1}$ be a t_X-determination sequence that is $\underset{\wedge}{X}$-wise chained, i.e., the sequence is $\underset{\wedge}{X}$-wise compact and for each $\underline{i} = 2,\ldots,\underline{m}$, $\underset{\wedge}{X}^o_i$ t_X-precedes $\underset{\wedge}{X}^o_{i+1}$ with $\underset{\wedge}{X}^o_{i+1}$ $\underset{\wedge}{X}$-wise solid. Then $\underset{\wedge}{X}^o_2;\ldots,\underset{\wedge}{X}^o_{m+1}$ is $\underset{\wedge}{X}$-wise well-ordered, and $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is a cd-series that is not only standard but $\underset{\wedge}{X}$-wise strongly compact.

<u>Proof</u>. Assume the theorem's preconditions and hypothesize for disproof that some variable $\underset{\wedge}{x}$ in any $\underset{\wedge}{X}^{oo}_{i+1}$ is either in $\underset{\wedge}{X}^o_i$ or is an $\underset{\wedge}{X}$-wise source of some variable in $\underset{\wedge}{X}^o_i$. Since this $\underset{\wedge}{x}$ in $\underset{\wedge}{X}^{oo}_{i+1}$ must be in some $\underset{\wedge}{X}^o_j$ with $\underline{j} \geq \underline{i}+1$, and by transitivity of t_X-precedence $\underset{\wedge}{X}^o_i$ t_X-precedes $\underset{\wedge}{X}^o_{j-1}$ which in turn t_X-precedes $\underset{\wedge}{X}^o_j$, there would then be a sequence $\langle \underset{\wedge}{x}, \underset{\wedge}{x}_i, \underset{\wedge}{x}_{j-1}, \underset{\wedge}{x}_j \rangle$ of variables wherein $\underset{\wedge}{x}$ is either identical with or is an $\underset{\wedge}{X}$-wise source of $\underset{\wedge}{x}_i$ which is in $\underset{\wedge}{X}^o_i$, $\underset{\wedge}{x}_i$ is either identical with $\underset{\wedge}{x}_{j-1}$ (if $\underline{j} = \underline{i}+1$) or is an $\underset{\wedge}{X}$-wise source of $\underset{\wedge}{x}_{j-1}$ which is in $\underset{\wedge}{X}^o_{j-1}$, and $\underset{\wedge}{x}_{j-1}$ is an $\underset{\wedge}{X}$-wise source of $\underset{\wedge}{x}_j$ which is in $\underset{\wedge}{X}^o_j$. With $\underset{\wedge}{x}$ and $\underset{\wedge}{x}_j$ both in $\underset{\wedge}{X}^o_j$, there would thus be a path from $\underset{\wedge}{X}^o_j$ to $\underset{\wedge}{X}^o_j$ that includes $\underset{\wedge}{x}_{j-1}$, whence by solidity of $\underset{\wedge}{X}^o_j$, $\underset{\wedge}{x}_{j-1}$ would be in $\underset{\wedge}{X}^o_j$--which is impossible, since $\underset{\wedge}{x}_{j-1}$ is in $\underset{\wedge}{X}_{j-1}$ which is disjoint from $\underset{\wedge}{X}^o_j$. So for all $\underline{i} = 2,\ldots,\underline{m}$, $\underset{\wedge}{X}^o_i$ must be $\underset{\wedge}{X}$-wise c-independent of $\underset{\wedge}{X}^{oo}_{i+1}$, which is to say that $\underset{\wedge}{X}^o_2;\ldots;\underset{\wedge}{X}^o_{m+1}$ is $\underset{\wedge}{X}$-wise well-ordered. From there and the sequence's stipulated $\underset{\wedge}{X}$-wise compactness, conclusion that $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is a standard cd-series that is $\underset{\wedge}{X}$-wise strongly compact is immediate from Theorem 19. □

The precession of stages in an $\underset{\wedge}{X}$-wise chained t_X-determination sequence can be continued just as long as the precession of omissions ordered by t_X-precedence can be continued with interior variables of $\underset{\wedge}{X}$. Specifically, for any t_X-determination sequence $\underset{\wedge}{X}_i \Rightarrow \underset{\wedge}{X}_{i+1} \Rightarrow \dots \Rightarrow \underset{\wedge}{X}_{m+1}$ that is $\underset{\wedge}{X}$-wise chained, let $\underset{\wedge}{X}''_i$ comprise the variables in $\underset{\wedge}{X}_i$ that are both in $\underline{I}(\underset{\wedge}{X})$ and are $\underset{\wedge}{X}$-wise sources of $\underset{\wedge}{X}^o_{i+1}$. It may be that $\underset{\wedge}{X}''_i$ is null; for although $\underset{\wedge}{X}_i$ includes at least some of the $\underset{\wedge}{X}$-wise direct sources of $\underset{\wedge}{X}^o_{i+1}$, these may all be in $\underline{E}(\underset{\wedge}{X})$. But if $\underset{\wedge}{X}''_i$ is not null, it contains one or more $\underset{\wedge}{X}$-wise solid subtuples (singletons, if no other), any of which t_X-precedes $\underset{\wedge}{X}^o_{i+1}$ and can be taken for $\underset{\wedge}{X}^o_i$. Then if $\underset{\wedge}{X}_{i-1} \doteq \langle \underset{\wedge}{X}'_{i-1}, \underset{\wedge}{X}_i\text{-not-}\underset{\wedge}{X}^o_i \rangle$ where $\underset{\wedge}{X}'_i$ is any tuple of $\underset{\wedge}{X}$-variables disjoint from $\underset{\wedge}{X}^o_i$ that t_X-determines $\underset{\wedge}{X}^o_i$ while including all $\underset{\wedge}{X}$-wise direct sources of $\underset{\wedge}{X}^o_i$ not already in $\underset{\wedge}{X}_i$, $\underset{\wedge}{X}_{i-1};\underset{\wedge}{X}_i;\dots;\underset{\wedge}{X}_{m+1}$ too is an $\underset{\wedge}{X}$-wise chained t_X-determination sequence. (At least one such $\underset{\wedge}{X}'_{i-1}$ exists because all variables in $\underset{\wedge}{X}^o_i$ are interior to $\underset{\wedge}{X}$.) Note further that whether $\underset{\wedge}{X}_{i-1}$ continues the chain's precession is judged just from the $\underset{\wedge}{X}$-wise causal relations among $\underset{\wedge}{X}_{i-1}$, $\underset{\wedge}{X}_i$, and $\underset{\wedge}{X}_{i+1}$ without consideration of stages after $\underset{\wedge}{X}_{i+1}$. Chained t_X-determination sequences are identified just by local structure in the sense that any sequence $\underset{\wedge}{X}_1;\dots;\underset{\wedge}{X}_{m+1}$ is an $\underset{\wedge}{X}$-wise chained t_X-determination sequence just in case, for each $\underline{i} = 2,\dots,\underline{m}$, $\underset{\wedge}{X}_{i-1};\underset{\wedge}{X}_i;\underset{\wedge}{X}_{i+1}$ is an $\underset{\wedge}{X}$-wise chained t_X-determination sequence.

When the precession of stages in an $\underset{\wedge}{X}$-wise chained t_X-determination sequence $\underset{\wedge}{X}_i \Rightarrow \underset{\wedge}{X}_{i+1} \Rightarrow \dots \Rightarrow \underset{\wedge}{X}_{m+1}$ has been continued as far as possible, i.e. when $\underset{\wedge}{X}_i$ contains no $\underset{\wedge}{X}$-wise sources of $\underset{\wedge}{X}^o_{i+1}$ that have $\underset{\wedge}{X}$-wise sources of their own, $\underset{\wedge}{X}_i$ will in general still contain variables in $\underline{I}(\underset{\wedge}{X})$ that can be replaced by some t_X-determiner $\underset{\wedge}{X}'_{i-1}$ thereof and so extend the precession even though the extended sequence is no longer $\underset{\wedge}{X}$-wise chained. But even then there may not exist any continuation stage $\underset{\wedge}{X}_{i-1}$ that preserves the sequence's character as a standard cd-series. To continue the precession of a standard cd-series' stages until all variables interior to background tuple $\underset{\wedge}{X}$ have been replaced by their sources in $\underline{E}(\underset{\wedge}{X})$, we need t_X-determination sequences that are $\underset{\wedge}{X}$-wise solidly conservative.

<u>Theorem</u> 21. Let $\underset{\wedge}{X}_1 \Rightarrow \ldots \Rightarrow \underset{\wedge}{X}_{m+1}$ be a t_X-determination sequence that is $\underset{\wedge}{X}$-wise **solidly conservative** (cf. Def. 2.24-2). Then ($\underline{a}$) for each $\underline{i} = 1,\ldots,\underline{m}$, $\underset{\wedge}{X}_i$ t_X-precedes $\underset{\wedge}{X}^{o}_{m+1}$ and is $\underset{\wedge}{X}$-wise causally independent of $\underset{\wedge}{X}^{oo}_{i+1}$, ($\underline{b}$) $\underset{\wedge}{X}^{o}_2;\ldots;\underset{\wedge}{X}^{o}_{m+1}$ is $\underset{\wedge}{X}$-wise well-ordered, and ($\underline{c}$) $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is a standard and $\underset{\wedge}{X}$-wise strongly compact cd-series wherein each $\underset{\wedge}{X}^{o}_{i+1}$ ($\underline{i} = 1,\ldots,\underline{m}$) is $\underset{\wedge}{X}$-wise solid.

<u>Proof</u>. Assume the theorem's preconditions. Since each $\underset{\wedge}{X}_i$ t_X-determines $\underset{\wedge}{X}_{i+1}$ ($\underline{i} = 1,\ldots,\underline{m}$), each variable $\underset{\wedge}{x}$ in $\underset{\wedge}{X}_i$ is either identical with or is an $\underset{\wedge}{X}$-wise source of some variable $\underset{\wedge}{x}'$ in $\underset{\wedge}{X}_m$ and hence not in $\underset{\wedge}{X}^{o}_{m+1}$. If $\underset{\wedge}{x}'$ is not in $\underset{\wedge}{X}_{m+1}$, $\underset{\wedge}{x}'$ and hence $\underset{\wedge}{x}$ is an $\underset{\wedge}{X}$-wise source of some variable in $\underset{\wedge}{X}^{o}_{m+1}$. Whereas if $\underset{\wedge}{x}'$ is in $\underset{\wedge}{X}_{m+1}$ it is in $\underset{\wedge}{X}_{m+1}$-not-$\underset{\wedge}{X}^{o}_{m+1}$, whence $\underset{\wedge}{x}'$ and hence $\underset{\wedge}{x}$ is again an $\underset{\wedge}{X}$-wise source of some variable in $\underset{\wedge}{X}^{o}_{m+1}$ by the constraint on $\underset{\wedge}{X}_{m+1}$ in the definition of solid conservatism. So each $\underset{\wedge}{X}_i$ ($\underline{i} < \underline{m}+1$) t_X-precedes $\underset{\wedge}{X}^{o}_{m+1}$ as claimed first in the theorem. Next, we show that for each $\underline{i} = 1,\ldots,\underline{m}$, no variable $\underset{\wedge}{x}$ in $\underset{\wedge}{X}_i$ is either in or has an $\underset{\wedge}{X}$-wise source in $\underset{\wedge}{X}^{oo}_{i+1}$. If $\underset{\wedge}{x}$ did have an $\underset{\wedge}{X}$-wise source in $\underset{\wedge}{X}^{oo}_{i+1}$, since $\underset{\wedge}{x}$ is an $\underset{\wedge}{X}$-wise source of some variable in $\underset{\wedge}{X}^{o}_{m+1}$ it would follow by solidity of $\underset{\wedge}{X}^{oo}_{i+1}$ that $\underset{\wedge}{x}$ is in the latter; hence it only remains to disprove that $\underset{\wedge}{x}$ is in $\underset{\wedge}{X}^{oo}_{i+1}$. Suppose to the contrary that $\underset{\wedge}{x}$ is not only in $\underset{\wedge}{X}_i$ but also in $\underset{\wedge}{X}^{o}_j$ for some $\underline{j} > \underline{i}$. Then $\underline{j} \geq \underline{i}+2$ because $\underset{\wedge}{X}_i$ is disjoint from $\underset{\wedge}{X}^{o}_{i+1}$. And $\underset{\wedge}{x}$ cannot be in $\underset{\wedge}{X}_{j-1}$ (since this is disjoint from $\underset{\wedge}{X}^{o}_j$), so by virtue of being in $\underset{\wedge}{X}_i$, which t_X-determines $\underset{\wedge}{X}_{j-1}$, $\underset{\wedge}{x}_i$ must be an $\underset{\wedge}{X}$-wise source of some variable $\underset{\wedge}{x}'$ in $\underset{\wedge}{X}_{j-1}$ that in turn is an $\underset{\wedge}{X}$-wise source of some variable in $\underset{\wedge}{X}^{o}_{m+1}$, whence by the solidity of $\underset{\wedge}{X}^{oo}_j$ this $\underset{\wedge}{x}'$ in $\underset{\wedge}{X}_{j-1}$ is also in $\underset{\wedge}{X}^{oo}_j$. But then $\underset{\wedge}{X}_{j-1}$ is not $\underset{\wedge}{X}$-wise c-independent of $\underset{\wedge}{X}^{oo}_j$--which is to say that $\underset{\wedge}{X}_i$ fails to be $\underset{\wedge}{X}$-wise c-independent of $\underset{\wedge}{X}^{oo}_{i+1}$ only if, for some $\underline{k} > \underline{i}$, $\underset{\wedge}{X}_k$ is not $\underset{\wedge}{X}$-wise c-independent of $\underset{\wedge}{X}^{oo}_{k+1}$. From there it is a simple conclusion by induction that for each $\underline{i} = \underline{m},\underline{m}-1,\ldots,1$, $\underset{\wedge}{X}_i$ is $\underset{\wedge}{X}$-wise c-independent of $\underset{\wedge}{X}^{oo}_{i+1}$. And since $\underset{\wedge}{X}^{o}_i$ is a subtuple of $\underset{\wedge}{X}_i$, each $\underset{\wedge}{X}^{o}_i$ too is $\underset{\wedge}{X}$-wise c-independent of $\underset{\wedge}{X}^{oo}_{i+1}$ --which from Theorem 19 and the compactness included in the definition of solid conservatism yields that $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is a standard and $\underset{\wedge}{X}$-wise strongly compact cd-series. Finally, that each $\underset{\wedge}{X}_i$ ($\underline{i} = 2,\ldots,\underline{m}+1$) is $\underset{\wedge}{X}$-wise solid follows directly from $\underset{\wedge}{X}$-wise solidity of $\underset{\wedge}{X}^{oo}_i$ together with the $\underset{\wedge}{X}$-wise c-independence of $\underset{\wedge}{X}^{o}_i$ from $\underset{\wedge}{X}^{oo}_{i+1}$. □

Although Theorem 21 does not have the macrostructural importance of Theorem 20, it is nevertheless of interest as a molar counterpart of the microstructural point, noted previously on p. 2.17 and in Theorem 5, that when single variables are sequentially eliminated from a given microcausal path structure in inverse order of causal independence, the variables that remain retain the same proximal sources before and after each reduction step. An $\underset{\wedge}{X}$-wise solidly conservative t_X-determination sequence $\ldots \Rightarrow \underset{\wedge}{X}_i \Rightarrow \ldots \Rightarrow \underset{\wedge}{X}_{m+1}$ is in effect constructed as follows: At each precession stage $\underset{\wedge}{X}_i$, consider the tuple $\underset{\wedge}{X}^*_i$ comprising all variables in $\underline{I}(\underset{\wedge}{X})$-not-$\underset{\wedge}{X}^{oo}_{i+1}$ that are $\underset{\wedge}{X}$-wise sources of $\underset{\wedge}{X}^{o}_{m+1}$. By the strict-partial-order character of the $\underset{\wedge}{X}$-wise source relation, if $\underset{\wedge}{X}^*_i$ is not null there is at least one variable $\underset{\wedge}{x}$ in $\underset{\wedge}{X}^*_i$ that is not an $\underset{\wedge}{X}$-wise source of any other variable in $\underset{\wedge}{X}^*_i$, and is hence an $\underset{\wedge}{X}$-wise direct source of some variable in $\underset{\wedge}{X}^{oo}_{i+1}$. Moreover, this $\underset{\wedge}{x}$ must be in $\underset{\wedge}{X}_i$, since by compactness each $\underset{\wedge}{X}$-wise direct source of any variable in $\underset{\wedge}{X}^{oo}_{i+1}$ is in $\underset{\wedge}{X}_i$ if it is not in $\underset{\wedge}{X}^{oo}_{i+1}$. And $\langle \underset{\wedge}{x}, \underset{\wedge}{X}^{oo}_{i+1} \rangle$ must be $\underset{\wedge}{X}$-wise solid, since $\underset{\wedge}{x}$ is $\underset{\wedge}{X}$-wise c-independent of $\underset{\wedge}{X}^{oo}_{i+1}$ and no path in $\underset{\wedge}{X}$ from $\underset{\wedge}{x}$ to $\underset{\wedge}{X}^{oo}_{i+1}$ can include a variable not in $\langle \underset{\wedge}{x}, \underset{\wedge}{X}^{oo}_{i+1} \rangle$ without violating x's status as a variable of which every other variable in $\underset{\wedge}{X}^*_i$ is $\underset{\wedge}{X}$-wise c-independent. So if $\underset{\wedge}{X}^*_i$ is not null, there is at least one non-null subtuple $\underset{\wedge}{X}^{o}_i$ of $\underset{\wedge}{X}_i$ (possibly but not necessarily a singleton) that contains just variables in $\underline{I}(\underset{\wedge}{X})$ and for which $\langle \underset{\wedge}{X}^{o}_i, \underset{\wedge}{X}^{oo}_{i+1} \rangle$ is $\underset{\wedge}{X}$-wise solid. From there, putting $\underset{\wedge}{X}_{i-1} = \langle \underset{\wedge}{X}'_{i-1}, \underset{\wedge}{X}_i\text{-not-}\underset{\wedge}{X}^{o}_i \rangle$ for some $\underset{\wedge}{X}'_{i-1}$ that is disjoint from but t_X-determines $\underset{\wedge}{X}^{o}_i$ while including all $\underset{\wedge}{X}$-wise direct sources of $\underset{\wedge}{X}^{o}_i$ that are not already in $\underset{\wedge}{X}^{o}_i$ gives $\underset{\wedge}{X}_{i-1};\underset{\wedge}{X}_i;\ldots;\underset{\wedge}{X}_{m+1}$ to be an extension of the t_X-determination precession that preserves its $\underset{\wedge}{X}$-wise solid conservatism.

Finally, to close our present study of composable determination sequences, there is an especially strong variety of standard cd-series, foreshadowed in Theorem 6, that also merits explicit recognition. For convenience, say

<u>Definition</u> 2.25. A tuple $\underset{\wedge}{X}$ is (<u>causally</u>) <u>thin</u> iff $\underset{\wedge}{X}$ has null interior. A sequence $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ of tuples is <u>essentially thin</u> iff each stage $\underset{\wedge}{X}_i$ prior to $\underset{\wedge}{X}_m$ therein is thin, i.e., iff $\underline{I}(\underset{\wedge}{X}_i)$ is null for all $\underline{i} = 1,\ldots,\underline{m}$-1.

Then,

<u>Theorem</u> 22. If $\underset{\wedge}{X} \overset{\bullet}{\Rightarrow} \underset{\wedge}{Y} \overset{\bullet}{\Rightarrow} \underset{\wedge}{Z}$ and $\underset{\wedge}{X}$ is thin, then $\underset{\wedge}{Y}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Z}$. <u>Corollary</u>. Any s-determination sequence that is essentially thin is a standard cd-series.

<u>Proof</u>. Suppose that $\underset{\wedge}{X}$, $\underset{\wedge}{Y}$, and $\underset{\wedge}{Z}$ are as stipulated. Then $\underset{\wedge}{Y}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Z}$ if conditions (<u>a</u>) and (<u>b</u>) of Theorem 14 are satisfied. Note also that $\underline{E}(\underset{\wedge}{X}) = \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y}) = \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y},\underset{\wedge}{Z}) = \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Z})$ while all variables in $\underset{\wedge}{Z}$-not-$\underset{\wedge}{X}$ are in $\underline{I}(\underset{\wedge}{X},\underset{\wedge}{Z})$ and hence in $\underline{I}(\underset{\wedge}{X},\underset{\wedge}{Y},\underset{\wedge}{Z})$ Any variable $\underset{\wedge}{z}_j$ common to $\underset{\wedge}{Z}$ and $\underset{\wedge}{X}$ must also be in $\underset{\wedge}{Y}$; since otherwise, were $\underset{\wedge}{z}_j$ to be in $\underset{\wedge}{Z}$-not-$\underset{\wedge}{Y}$, it would be in $\underline{I}(\underset{\wedge}{Y},\underset{\wedge}{Z})$ (by premise $\underset{\wedge}{Y} \overset{\bullet}{\Rightarrow} \underset{\wedge}{Z}$) and hence in $\underline{I}(\underset{\wedge}{X},\underset{\wedge}{Y},\underset{\wedge}{Z})$, whence $\underset{\wedge}{z}_j$ would also be in $\underline{I}(\underset{\wedge}{X})$ (since $\underset{\wedge}{z}_j$ is in $\underset{\wedge}{X}$ and $\underline{E}(\underset{\wedge}{X}) = \underline{E}(\underset{\wedge}{X},\underset{\wedge}{Y},\underset{\wedge}{Z})$) contrary to stipulation that $\underline{I}(\underset{\wedge}{X})$ is null. So condition (<u>a</u>) of Theorem 14 is satisfied. To see that condition (<u>b</u>) also holds, let $\underset{\wedge}{x}_i$ and $\underset{\wedge}{z}_j$ be any variables in $\underset{\wedge}{X}$-not-$\underset{\wedge}{Y}$ and $\underset{\wedge}{Z}$-not-$\underset{\wedge}{Y}$, respectively, and suppose that $\underset{\wedge}{W}_{ij}$ is any path in $\underset{\wedge}{W} =_{def} \langle \underset{\wedge}{X},\underset{\wedge}{Y},\underset{\wedge}{Z} \rangle$ from $\underset{\wedge}{x}_i$ to $\underset{\wedge}{z}_j$. Since $\underset{\wedge}{X}$ has null interior, $\underset{\wedge}{x}_i$ is in $\underline{E}(\underset{\wedge}{X})$ and hence in $\underline{E}(\underset{\wedge}{W})$; so $\underset{\wedge}{W}_{ij}$ is a total path to $\underset{\wedge}{z}_j$ in $\underset{\wedge}{W}$ starting with $\underset{\wedge}{x}_i$. Moreover, $\underset{\wedge}{x}_i$ is the only $\underset{\wedge}{X}$-variable in $\underset{\wedge}{W}_{ij}$, else some other $\underset{\wedge}{X}$-variable would be interior to $\underset{\wedge}{W}$ and hence to $\underset{\wedge}{X}$. So unless $\underset{\wedge}{W}_{ij}$ contains a variable in $\underset{\wedge}{Y}$, either $\underset{\wedge}{z}_j$ is not interior to $\underset{\wedge}{W}$-not-$\underset{\wedge}{x}_i$ (which occurs if $\underset{\wedge}{x}_i$ is a direct source in $\underset{\wedge}{W}$ of all other variables in $\underset{\wedge}{W}_{ij}$) or some terminal segment of $\underset{\wedge}{W}_{ij}$ containing only variables in $\underset{\wedge}{Z}$-not-$\underset{\wedge}{Y}$ is a total path to $\underset{\wedge}{z}_j$ in $\underset{\wedge}{W}$-not-$\underset{\wedge}{x}_i$. Either way, failure of $\underset{\wedge}{W}_{ij}$ to pass through $\underset{\wedge}{Y}$ entails that some variable in $\underset{\wedge}{Z}$-not-$\underset{\wedge}{Y}$ is in the exterior of $\underset{\wedge}{W}$-not-$\underset{\wedge}{x}_i$--which is impossible, since every variable in $\underset{\wedge}{Z}$-not-$\underset{\wedge}{Y}$ is interior to $\langle \underset{\wedge}{Y},\underset{\wedge}{Z} \rangle$ and hence (since $\underset{\wedge}{x}_i$ is not in $\underset{\wedge}{Y}$ or $\underset{\wedge}{Z}$) interior to $\underset{\wedge}{W}$-not-$\underset{\wedge}{x}_i$ = $\langle \underset{\wedge}{X}$-not-$\underset{\wedge}{x}_i,\underset{\wedge}{Y},\underset{\wedge}{Z} \rangle$. So every path from $\underset{\wedge}{X}$-not-$\underset{\wedge}{Y}$ to $\underset{\wedge}{Z}$-not-$\underset{\wedge}{Y}$ must pass through $\underset{\wedge}{Y}$. The corollary is immediate from Def. 2.19. □

In view of Theorem 22, thinness is an extremely attractive property for Tuples to have, one under which the causal composability of an s-determination sequence's single-step transducers can be diagnosed from just the local structure of each constituent tuple considered apart from all the others (together of course with the s-determinacy between tuples adjacent in the sequence), Moreover, any s-determination sequence $\mathbf{X}_1 \doteq\!> \ldots \doteq\!> \mathbf{X}_{m+1}$ can always be reduced to an essentially thin one with the same terminal stage $\mathbf{X}_{m+1}$ simply by replacing each $\mathbf{X}_i$ prior to $\mathbf{X}_{m+1}$ therein by $\underline{E}(\mathbf{X}_i)$. (Replacement of $\mathbf{X}_m$ by $\underline{E}(\mathbf{X}_m)$ is optional.) However, if $\mathbf{X}_1 \doteq\!> \ldots \doteq\!> \mathbf{X}_{m+1}$ is a cd-series that is not essentially thin, its reduction $\underline{E}(\mathbf{X}_1) \doteq\!> \underline{E}(\mathbf{X}_2) \doteq\!> \ldots \doteq\!> \underline{E}(\mathbf{X}_m) \doteq\!> \mathbf{X}_{m+1}$ to essential thinness is not compositionally equivalent to the original sequence, insomuch as the causal transducers involved are nontrivially different. Specifically, if $\mathbf{Y} \doteq\!> \mathbf{Z}$ (where we take $\mathbf{Y}$ for any $\mathbf{X}_i$ and $\mathbf{Z}$ for any later $\mathbf{X}_j$ in the series), the $\underline{k}$th component $\mathbf{z}_k = \phi_k(\mathbf{Y})$ of the quasi-causal regularity under which $\mathbf{Y}$ determines $\mathbf{Z}$ generally fails to embed the causal regularity or noncausal identity-selection $\mathbf{z}_k = \phi'_k(\underline{E}(\mathbf{Y}))$ under which $\underline{E}(\mathbf{Y})$ determines $\mathbf{z}_k$.

To be sure, given a cd-series $\mathbf{X}_1 \doteq\!> \ldots \doteq\!> \mathbf{X}_{m+1}$ that is not essentially thin, it may be possible to reduce this to one that is while preserving the essentials of the original series' transducers. The technique for this is to replace first $\mathbf{X}_m$ by its subtuple $\mathbf{X}'_m$ that contains only variables that are either in $\mathbf{X}_{m+1}$ or are a direct source within $\langle \mathbf{X}_m, \mathbf{X}_{m+1} \rangle$ of some variable in $\mathbf{X}_{m+1}$-not-$\mathbf{X}_m$, then to replace $\mathbf{X}_{m-1}$ by its subtuple $\mathbf{X}'_{m-1}$ containing only variables that are in $\mathbf{X}'_m$ or are a direct source in $\langle \mathbf{X}_{m-1}, \mathbf{X}'_m \rangle$ of some variable in $\mathbf{X}'_m$-not-$\mathbf{X}_{m-1}$, and so on recursively for $\underline{i} = \underline{m}, \underline{m}-1, \ldots, 1$. However, these reduced sequence stages $\mathbf{X}'_i$ are by no means certain to be thin in principle even though that may be a not-unreasonable assumption in most applied contexts. Considerably more remains to be said about this matter. But more is not called for on this occasion.

Partial compositions.

Although we have now examined the theory of composable macro-causal regularities in considerable detail, the situation just studied--s-determination sequences in which each stage is a complete quasi-source of all variables in its successor--is still not the most general form of macrocausal composition. Microstructurally, the problem of causal composability arises primarily from mediations wherein the output variable of one causal regularity is just one of the conjoint input variables in a second. Correspondingly, study of mediation at the molar level wants also to consider how quasi-causal regularity $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ can be composed into quasi-causal regularity $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z})$ when $\underset{\wedge}{Z}'$ comprises only some of the variables in $\underset{\wedge}{Z}$. When need for the distinction arises, we may call the latter case "partial" composition in contrast to "total" compositions in which the output tuple of one composing quasi-causal regularity is essentially identical with the total input to the other. Technically, however, it is most convenient to understand "partial composition" in a generic sense that subsumes total compositions as the limiting case wherein $\underset{\wedge}{Z}' \doteq \underset{\wedge}{Z}$.

Using the notation explained on p. 2.18f., the partial composition of $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ into $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z})$ when all $\underset{\wedge}{Z}'$-variables are in $\underset{\wedge}{Z}$ is $\underset{\wedge}{Y} = \phi\rho(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \psi'(\underset{\wedge}{X}))$, wherein ρ^{-1} is the permutation operator that rearranges $\underset{\wedge}{Z}$ as $\langle \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{Z}' \rangle$. To avoid needless complications, we shall assume that ρ is an Identity permutation, i.e., that $\underset{\wedge}{Z} = \langle Z\text{-not-}Z', Z' \rangle$ so that the composition at issue is just $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \psi'(\underset{\wedge}{X}))$. Given that these composing regularities are quasi-causal, we want to know the conditions under which their partial composition is also quasi-causal. The answer is of course already implicit in <u>CmP</u>-4 and Theorem 7. But it takes considerable effort to translate these microcausal principles into perspicuous molar terms. Happily, the bulk of that work has already been accomplished in Theorem 15 for total molar compositions. It only remains to show how the latter can be extended to cover partial compositions as well.

The extension is really quite simple. When $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z})$ and $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ are quasi-causal with all $\underset{\wedge}{Z}'$-variables in $\underset{\wedge}{Z}$, and for simplicity $\underset{\wedge}{Z}$ is ordered as $\underset{\wedge}{Z} = \langle \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{Z}' \rangle$, we have $\underset{\wedge}{X} \stackrel{\cdot}{\Rightarrow} \underset{\wedge}{Z}'$, $\underset{\wedge}{Z} \stackrel{\cdot}{\Rightarrow} \underset{\wedge}{Y}$, and hence $\langle \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X} \rangle \stackrel{\cdot}{\Rightarrow} \langle \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{Z}' \rangle \doteq \underset{\wedge}{Z} \stackrel{\cdot}{\Rightarrow} \underset{\wedge}{Y}$. That is, $\langle \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X} \rangle$ s-determines $\underset{\wedge}{Y}$, through the partial mediation of $\underset{\wedge}{Z}'$, under some quasi-causal regularity $\underset{\wedge}{Y} = \theta(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X})$. So the partial composition $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \psi'(\underset{\wedge}{X}))$ of $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ into $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z})$ is quasi-causal just in case this composition's transducer is θ. Let us assume that $\underset{\wedge}{Z}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$, since by Theorem 7 this is for all practical purposes a necessary condition for the partial composition to preserve causality. (Making clear how Theorem 7 has this molar implication is somewhat tedious, and will not be attempted here.) This is equivalent to presuming that $\underset{\wedge}{Z}$ disconnects $\langle \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X}\rangle$ from $\underset{\wedge}{Y}$. Then if $\underset{\wedge}{Z} = \psi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X})$ is the quasi-causal regularity under which $\langle \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X}\rangle$ determines $\underset{\wedge}{Z}$, Theorem 15 entails that $\underset{\wedge}{Y} = \phi\psi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X})$ is also quasi-causal, i.e., that $\theta = \phi\psi$. So given this disconnection premise, $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \psi'(\underset{\wedge}{X}))$ is quasi-causal just in case its transducer is $\phi\psi$. Finally, this partial composition's transducer is indeed $\phi\psi$ if and, with few if any significant exceptions, only if $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ is [redacted] embedded (cf. Def. 2.17) in $\underset{\wedge}{Z} = \psi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X})$. (I can't find any simple way to verbalize why that is so. One just has to think through the formalisms and <u>see</u> (<u>i</u>) that the transducers in $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \psi'(\underset{\wedge}{X}))$ and $\underset{\wedge}{Y} = \phi\psi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X})$ are both functions on the logical range of $\langle \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X}\rangle$; (<u>ii</u>) that $\langle \underline{Z}\text{-not-}\underline{Z}', \psi'(\underline{X})\rangle = \psi(\underline{Z}\text{-not-}\underline{Z}', \underline{X})$ for all arguments of these compositions just in case $\underset{\wedge}{Z} = \psi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X})$ embeds $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$; and (<u>iii</u>) that for any functions α and β whose values are arguments of ϕ, $\phi\alpha = \phi\beta$ if and, for all practical purposes, only if $\alpha = \beta$. To get clear on (<u>ii</u>), it must be understood both that if the <u>i</u>th variable $\underset{\wedge}{z}_i$ in $\underset{\wedge}{Z}$ is in $\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}'$ then the <u>i</u>th component function in $\underset{\wedge}{Z} = \psi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X})$ is an identity selector that picks $\underset{\wedge}{z}_i$ out of $\langle \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X}\rangle$, and that $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ is embedded in $\underset{\wedge}{Z} = \psi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X})$ just in case each component function in the latter for a variable in $\underset{\wedge}{Z}'$ differs from the function in the former for that same variable only in including variables in $\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}'$ with null weights.) So Theorem 15 also implicitly covers partial composition in the sense that

<u>Theorem</u> 23. If $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z})$ and $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ are quasi-causal regularities wherein $\underset{\wedge}{Z}$ includes all variables in $\underset{\wedge}{Z}'$, the (partial) composition of $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ into $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z})$ is also quasi-causal if and, for the most part, only if $\underset{\wedge}{Z}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$ and the quasi-causal regularity under which $\langle \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X}\rangle$ determines $\underset{\wedge}{Z}$ embeds $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$. (Note that if $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ is embedded in $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z})$, the embedding is pre-emptive.)

The structural conditions that satisfy Theorem 23's embedding requirement are straightforward from the definition of embedding: Given that these regularities are both quasi-causal with all $\underset{\wedge}{Z}'$-variables in $\underset{\wedge}{Z}$, $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ is embedded in $\underset{\wedge}{Z} = \psi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X})$ just in case, for each variable $\underset{\wedge}{z}'_i$ in $\underset{\wedge}{Z}'$, either $\underset{\wedge}{z}'_i$ is in $\underset{\wedge}{X}$ (in which case both $\underset{\wedge}{Z} = \psi(\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X})$ and $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ determine $\underset{\wedge}{z}_i$ by noncausal identity-selection from their respective input tuples) or $\underset{\wedge}{z}'_i$ has the same proximal source within $\underset{\wedge}{X}$ as it has within $\langle \underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}', \underset{\wedge}{X} \rangle$. This either/or condition for the embedding holds for all variables in $\underset{\wedge}{Z}'$ just in case all paths from $\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}'$ to $\underset{\wedge}{Z}'$ within $\langle \underset{\wedge}{Z}, \underset{\wedge}{X} \rangle$ pass through $\underset{\wedge}{X}$. And since $\underset{\wedge}{X} \stackrel{\cdot}{\Rightarrow} \underset{\wedge}{Z}'$ with $\underset{\wedge}{Z}'$ disjoint from $\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}'$, the latter is in turn equivalent (cf. Theorem 14) to saying that $\underset{\wedge}{X}$ disconnects $\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}'$ from $\underset{\wedge}{Z}'$. So Theorem 23 can be rewritten as

> __Theorem__ 23a. If $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z})$ and $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ are quasi-causal regularities wherein $\underset{\wedge}{Z}$ includes all variables in $\underset{\wedge}{Z}'$, the (partial) composition of $\underset{\wedge}{Z}' = \psi'(\underset{\wedge}{X})$ into $\underset{\wedge}{Y} = \phi(\underset{\wedge}{Z})$ is also quasi-causal if and, for the most part, only if $\underset{\wedge}{Z}$ disconnects $\underset{\wedge}{X}$ from $\underset{\wedge}{Y}$ and $\underset{\wedge}{X}$ disconnects $\underset{\wedge}{Z}\text{-not-}\underset{\wedge}{Z}'$ from $\underset{\wedge}{Z}'$.

It only remains to show how partial composition works out in development of cd-series. Recall that any s-determination sequence $\underset{\wedge}{X}_1 \stackrel{\cdot}{\Rightarrow} \ldots \stackrel{\cdot}{\Rightarrow} \underset{\wedge}{X}_{m+1}$ can be viewed as a precession $\underset{\wedge}{X}_i \stackrel{\cdot}{\Rightarrow} \underset{\wedge}{X}_{i+1}$ ($\underline{i} = m, \underline{m}-1, \ldots, 1$) in which at each stage a subtuple $\underset{\wedge}{X}^o_{i+1}$ of $\underset{\wedge}{X}_{i+1}$ is replaced by an s-determiner $\underset{\wedge}{X}'_i$ thereof with $\underset{\wedge}{X}'_i$ and $\underset{\wedge}{X}^o_{i+1}$ disjoint, i.e., $\underset{\wedge}{X}^o_{i+1} = \underset{\wedge}{X}_{i+1}\text{-not-}\underset{\wedge}{X}_i$, $\underset{\wedge}{X}'_i \stackrel{\cdot}{\Rightarrow} \underset{\wedge}{X}^o_{i+1}$, and $\underset{\wedge}{X}_i \doteq \langle \underset{\wedge}{X}'_i, \underset{\wedge}{X}_{i+1}\text{-not-}\underset{\wedge}{X}^o_{i+1} \rangle$. If $\underset{\wedge}{X}^o_{i+1} = \phi'_i(\underset{\wedge}{X}'_i)$ is the quasi-causal regularity by which $\underset{\wedge}{X}'_i$ determines $\underset{\wedge}{X}^o_{i+1}$, when does that suffice to identify the regularity $\underset{\wedge}{X}_{i+1} = \phi_i(\underset{\wedge}{X}_i)$ under which all of $\underset{\wedge}{X}_i$ s-determines all of $\underset{\wedge}{X}_{i+1}$? This identification obtains just in case $\underset{\wedge}{X}^o_{i+1} = \phi'_i(\underset{\wedge}{X}'_i)$ is pre-emptively embedded in $\underset{\wedge}{X}_{i+1} = \phi_i(\underset{\wedge}{X}_i)$, i.e., just in case the latter can be constructed from the other just by null-weight insertions of variables $\underset{\wedge}{X}_i\text{-not-}\underset{\wedge}{X}'_i$ into the determination of $\underset{\wedge}{X}^o_{i+1}$ by $\underset{\wedge}{X}'_i$, together with identity selectors to pick variables in $\underset{\wedge}{X}_{i+1}\text{-not-}\underset{\wedge}{X}^o_{i+1}$ out of $\underset{\wedge}{X}_i$. We shall now see that with only routine care in selecting $\underset{\wedge}{X}'_i$, this desired embedding always holds for, inter alia, sequences satisfying the preconditions of Theorems 20 & 21.

First, let us clarify how Theorem 23/23__a__ applies to a standard cd-series

$\underset{\wedge}{X}_1 \overset{\bullet}{\Rightarrow} \ldots \overset{\bullet}{\Rightarrow} \underset{\wedge}{X}_{m+1}$ of $\underset{\wedge}{X}$-subtuples. As before, $\underset{\wedge}{X}^o_{i+1} =_{def} \underset{\wedge}{X}_{i+1}$-not-$\underset{\wedge}{X}_i$, and we also presume that our interest in the s-determination of each $\underset{\wedge}{X}_{i+1}$ by $\underset{\wedge}{X}_i$ is focused on a distinguished subtuple $\underset{\wedge}{X}'_i$ of $\underset{\wedge}{X}_i$ that s-determines $\underset{\wedge}{X}^o_{i+1}$ while being disjoint from the latter. (Before we are done, $\underset{\wedge}{X}'_i$ will receive an additional constraint.) To subsume $\underset{\wedge}{X}_i \overset{\bullet}{\Rightarrow} \underset{\wedge}{X}_{i+1} \overset{\bullet}{\Rightarrow} \underset{\wedge}{X}_{i+2}$ under Theorem 23/23a, we take $\underset{\wedge}{X}_{i+2}$ for $\underset{\wedge}{Y}$, $\underset{\wedge}{X}_{i+1}$ for $\underset{\wedge}{Z}$, $\underset{\wedge}{X}^o_{i+1}$ for $\underset{\wedge}{Z}'$, and $\underset{\wedge}{X}'_i$ for $\underset{\wedge}{X}$, whence $\underset{\wedge}{Z}$-not-$\underset{\wedge}{Z}'$ becomes $\underset{\wedge}{X}_{i+1}$-not-$\underset{\wedge}{X}^o_{i+1}$ and some permutation of $\langle \underset{\wedge}{Z}$-not-$\underset{\wedge}{Z}', \underset{\wedge}{X}\rangle$ becomes $\underset{\wedge}{X}_i$. By stipulation that this cd-series is standard, $\underset{\wedge}{X}_{i+1}$ disconnects $\underset{\wedge}{X}'_i$ from $\underset{\wedge}{X}_{i+2}$. So if $\underset{\wedge}{X}_{i+2} = \phi_{i+1}(\underset{\wedge}{X}_{i+1})$, $\underset{\wedge}{X}_{i+1} = \phi_i(\underset{\wedge}{X}_i)$, and $\underset{\wedge}{X}^o_{i+1} = \phi'_i(\underset{\wedge}{X}'_i)$ are the quasi-causal regularities under which $\underset{\wedge}{X}_{i+1}$, $\underset{\wedge}{X}_i$, and $\underset{\wedge}{X}'_i$ respectively determine $\underset{\wedge}{X}_{i+2}$, $\underset{\wedge}{X}_{i+1}$, and $\underset{\wedge}{X}^o_{i+1}$, Theorem 23/23a tells us that the quasi-causal regularity $\underset{\wedge}{X}_{i+2} = \phi_{i+1}\phi_i(\underset{\wedge}{X}_i)$ under which $\underset{\wedge}{X}_i$ determines $\underset{\wedge}{X}_{i+2}$ through the mediation of $\underset{\wedge}{X}_{i+1}$ is logically equivalent to the partial composition of $\underset{\wedge}{X}^o_{i+1} = \phi'_i(\underset{\wedge}{X}'_i)$ into $\underset{\wedge}{X}_{i+2} = \phi'_{i+1}(\underset{\wedge}{X}_{i+1})$ if and for all practical purposes only if $\underset{\wedge}{X}^o_{i+1} = \phi'_i(\underset{\wedge}{X}'_i)$ is (pre-emptively) embedded in $\underset{\wedge}{X}_{i+1} = \phi_i(\underset{\wedge}{X}_i)$, i.e., if and essentially only if $\underset{\wedge}{X}'_i$ disconnects $\underset{\wedge}{X}_{i+1}$-not-$\underset{\wedge}{X}^o_{i+1}$ from $\underset{\wedge}{X}^o_{i+1}$. And since $(\underset{\wedge}{X}_{i+1}$-not-$\underset{\wedge}{X}^o_{i+1})$-not-$\underset{\wedge}{X}'_i = \underset{\wedge}{X}_i$-not-$\underset{\wedge}{X}'_i$, $\underset{\wedge}{X}'_i$ disconnects $\underset{\wedge}{X}_{i+1}$-not-$\underset{\wedge}{X}^o_{i+1}$ from $\underset{\wedge}{X}^o_{i+1}$ just in case X'_i disconnects X_i from X^o_{i+1}.

To obtain this disconnection under macrostructurally normal circumstances, let $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ be an $\underset{\wedge}{X}$-wise compact t_X-determination sequence that achieves standard cd-status through $\underset{\wedge}{X}$-wise well-ordering of $\underset{\wedge}{X}^o_2;\ldots;\underset{\wedge}{X}^o_{m+1}$ (cf. Theorem 19). Then for each $\underline{i} = 1,\ldots,\underline{m}$, all $\underset{\wedge}{X}$-wise direct sources of $\underset{\wedge}{X}^o_{i+1}$ are in $\langle \underset{\wedge}{X}_i, \underset{\wedge}{X}_{i+1}\rangle$ (= $\langle \underset{\wedge}{X}_i, \underset{\wedge}{X}^o_{i+1}\rangle$) by the strong-compactness consequence in Theorem 19. So without further constraining the $\underset{\wedge}{X}_i$ we can presume also that the $\underset{\wedge}{X}'_i$ part of each $\underset{\wedge}{X}_i$ has been chosen to t_X-determine $\underset{\wedge}{X}^o_{i+1}$ by including all $\underset{\wedge}{X}$-wise direct sources of $\underset{\wedge}{X}^o_{i+1}$ that are not in $\underset{\wedge}{X}^o_{i+1}$. The latter is equivalent to making each $\underset{\wedge}{X}'_i;\underset{\wedge}{X}^o_{i+1}$ $\underset{\wedge}{X}$-wise compact; and indeed, to attain the properties wanted for $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$, it suffices to stipulate that each $\underset{\wedge}{X}'_i$ t_X-precedes $\underset{\wedge}{X}^o_{i+1}$ with $\underset{\wedge}{X}'_i;\underset{\wedge}{X}^o_{i+1}$ $\underset{\wedge}{X}$-wise compact and $\underset{\wedge}{X}^o_2;\ldots;\underset{\wedge}{X}^o_{m+1}$ $\underset{\wedge}{X}$-wise well-ordered. Then $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ in its entirety is an $\underset{\wedge}{X}$-wise strongly compact t_X-determination sequence that is moreover a standard cd-series. Finally, let us also require each $\underset{\wedge}{X}^o_{i+1}$ ($\underline{i} = 1,\ldots,\underline{m}$) to be $\underset{\wedge}{X}$-wise solid, as holds for sequences to which Theorem 20 or Theorem 21 applies.

Then solidity of $\underset{\wedge}{X}^{o}_{i+1}$ combined with compactness of $\underset{\wedge}{X}'_{i};\underset{\wedge}{X}^{o}_{i+1}$ entails that every path in $\underset{\wedge}{X}$ from $\underset{\wedge}{X}_{i}$ to $\underset{\wedge}{X}^{o}_{i+1}$ has a terminal segment consisting of a variable in $\underset{\wedge}{X}'_{i}$ followed by one or more variables in $\underset{\wedge}{X}^{o}_{i+1}$; hence from Theorem 14, since $\underset{\wedge}{X}_{i}$ and $\underset{\wedge}{X}^{o}_{i+1}$ are disjoint, $\underset{\wedge}{X}'_{i}$ disconnects $\underset{\wedge}{X}_{i}$ from $\underset{\wedge}{X}^{o}_{i+1}$. In short,

<u>Theorem</u> 24. Let $\underset{\wedge}{X}_{1};\ldots;\underset{\wedge}{X}_{m+1}$ be a sequence of $\underset{\wedge}{X}$-subtuples assembled from the variables in quasi-causal regularities $\{\underset{\wedge}{X}^{o}_{i+1} = \phi'_{i}(\underset{\wedge}{X}'_{i})\}$ ($\underline{i} = 1,\ldots,\underline{m}$) and some possibly-null subtuple $\underset{\wedge}{X}^{+}_{m+1}$ of $\underset{\wedge}{X}$ in compliance with the following constraints: ($\underline{a}$) $\underset{\wedge}{X}^{o}_{2};\ldots;\underset{\wedge}{X}^{o}_{m+1}$ is $\underset{\wedge}{X}$-wise well-ordered with each $\underset{\wedge}{X}^{o}_{i}$ therein $\underset{\wedge}{X}$-wise solid. ($\underline{b}$) For each $\underline{i} = 1,\ldots,\underline{m}$, $\underset{\wedge}{X}'_{i}$ t_{X}-determines $\underset{\wedge}{X}^{o}_{i}$ with $\underset{\wedge}{X}'_{i}$ and $\underset{\wedge}{X}^{o}_{i+1}$ disjoint and $\underset{\wedge}{X}'_{i};\underset{\wedge}{X}^{o}_{i+1}$ X-wise compact. And ($\underline{c}$) $\underset{\wedge}{X}_{m+1} = \langle\underset{\wedge}{X}^{o}_{m+1},\underset{\wedge}{X}^{+}_{m+1}\rangle$, while for each $\underline{i} = 1,\ldots,\underline{m}$, $\underset{\wedge}{X}_{i} \doteq \langle\underset{\wedge}{X}'_{i},\underset{\wedge}{X}_{i+1}\text{-not-}X^{o}_{i+1}\rangle$, i.e., $\underset{\wedge}{X}_{i}$ comprises $\underset{\wedge}{X}'_{i}$ together with whatever $\underset{\wedge}{X}_{i+1}$-variables are not in $\underset{\wedge}{X}^{o}_{i+1}$. Then $\underset{\wedge}{X}_{1};\ldots;\underset{\wedge}{X}_{m+1}$ is a standard cd-series in which, for each $\underline{i} = 1,\ldots,\underline{m}$, $\underset{\wedge}{X}^{o}_{i+1} = \phi'_{i}(\underset{\wedge}{X}'_{i})$ is pre-emptively embedded in the quasi-causal regularity $\underset{\wedge}{X}_{i+1} = \phi_{i}(\underset{\wedge}{X}_{i})$ under which $\underset{\wedge}{X}_{i}$ determines $\underset{\wedge}{X}_{i+1}$. That is, $\underset{\wedge}{X}^{o}_{m+1} = \phi'_{m}(\underset{\wedge}{X}'_{m})$ is pre-emptively embedded in $\underset{\wedge}{X}_{m+1} = \phi_{m}(\underset{\wedge}{X}_{m})$ (and is identical with the latter in the paradigm case of null $\underset{\wedge}{X}^{+}_{m+1}$); the quasi-causal regularity $\underset{\wedge}{X}_{m+1} = \phi^{*}_{m-1}(\underset{\wedge}{X}_{m-1})$ under which $\underset{\wedge}{X}_{m-1}$ determines $\underset{\wedge}{X}_{m+1}$ through the partial mediation of $\underset{\wedge}{X}^{o}_{m}$ is the partial composition of $\underset{\wedge}{X}^{o}_{m} = \phi'_{m-1}(\underset{\wedge}{X}'_{m-1})$ into $\underset{\wedge}{X}_{m+1} = \phi_{m}(\underset{\wedge}{X}_{m})$; and more generally, recursively for $\underline{i} = \underline{m},\underline{m}-1,\ldots,1$, the quasi-causal regularity $\underset{\wedge}{X}_{m+1} = \phi^{*}_{i}(\underset{\wedge}{X}_{i})$ under which $\underset{\wedge}{X}_{i}$ determines $\underset{\wedge}{X}_{m+1}$ through the partial mediation of $\langle\underset{\wedge}{X}^{o}_{i+1},\ldots,\underset{\wedge}{X}^{o}_{m}\rangle$ is the partial composition of $\underset{\wedge}{X}^{o}_{i+1} = \phi'_{i}(\underset{\wedge}{X}'_{i})$ into $\underset{\wedge}{X}_{m+1} = \phi^{*}_{i+1}(\underset{\wedge}{X}_{i+1})$.

Conversely, whenever $\underset{\wedge}{X}_{1} \Rightarrow \ldots \Rightarrow \underset{\wedge}{X}_{m+1}$ is an $\underset{\wedge}{X}$-wise compact t_{X}-determination sequence in which $\underset{\wedge}{X}^{o}_{2};\ldots;\underset{\wedge}{X}^{o}_{m+1}$ is $\underset{\wedge}{X}$-wise well-ordered and each $\underset{\wedge}{X}^{o}_{i+1}$ ($\underline{i} = 1,\ldots,\underline{m}$) is $\underset{\wedge}{X}$-wise solid, notably when the preconditions of Theorem 20 or Theorem 21 are satisfied, the sequence is also strongly compact (cf. Theorem 19) so that subtuples $\underset{\wedge}{X}'_{i}$ of the $\underset{\wedge}{X}_{i}$ can be selected for which $\{\underset{\wedge}{X}'_{i}\}$, $\{\underset{\wedge}{X}_{i}\}$, and $\{\underset{\wedge}{X}^{o}_{i+1}\}$ satisfy the preconditions of Theorem 24. If we want, the total compositions for this cd-series $\underset{\wedge}{X}_{1};\ldots;\underset{\wedge}{X}_{m+1}$ in Theorem 16 fashion can be reformulated as an iteration of partial compositions as described in Theorem 24. But iterated partial compositions are difficult to handle conceptually. The **singular charm of standard** cd-series--a prime reason to think about causal relations in molar rather than microstructural terms--is that these allow us to formalize iterated partial compositions by linear strings of total compositions whose constituent quasi-causal transducers contain their strictly causal information in the form of embeddings.

<u>Molar path structure</u>.

Previously (p. 2.48) we reviewed the manifold aspects of microcausal structure represented by path digraphs. We are now in position to consider what a molar counterpart thereof might be.

Evidently, to be usefully isomorphic to its microcausal prototype, a macrocausal path digraph must comprise on the one hand a finite set $\Sigma_X = \{\underset{\wedge}{X}_i\}$ of Tuples, and on the other hand a partial-order relation $\xrightarrow{\theta}$ on Σ_X that directly or indirectly represents causal connection/mediation/disconnection/determination/composition relations among tuples in Σ_X in fashions corresponding as closely as we can manage to the microstructural path manifestations of these. To develop such an isomorphism, we can best seek first of all a molar counterpart for the microstructural model's most essential character, and then consider whether that gives us all we want or at least all that we can have.

The interpretively deepest feature of a microcausal path digraph $\mathcal{T}_X$, in which are joined all five facets of its representations, is that a path therein of length 2 or greater demarks the microcausal version of a chained cd-series. For, suppose that $\langle \underset{\wedge}{x}_1, \ldots, \underset{\wedge}{x}_{m+1} \rangle$ is a path from $\underset{\wedge}{x}_1$ to $\underset{\wedge}{x}_{m+1}$ within tuple $\underset{\wedge}{X}$. Then for each $\underline{i} = 1, \ldots, \underline{m}$, $\underset{\wedge}{x}_{i+1}$ has a proximal source $\underset{\wedge}{X}^*_i$ within $\underset{\wedge}{X}$ that includes $\underset{\wedge}{x}_i$. If we put $\underset{\wedge}{X}_{m+1} =_{def} \langle \underset{\wedge}{x}_{m+1} \rangle$ and $\underset{\wedge}{X}_i =_{def} \langle \underset{\wedge}{X}^*_i, \underset{\wedge}{X}_{i+1}\text{-not-}\underset{\wedge}{x}_{i+1} \rangle$ for $\underline{i} = \underline{m}, \underline{m}-1, \ldots, 1$, each $\underset{\wedge}{X}_i$ is formed by replacing $\underset{\wedge}{x}_{i+1}$ in $\underset{\wedge}{X}_{i+1}$ by $\underset{\wedge}{x}_i$ together with the other $\underset{\wedge}{X}$-wise direct sources of $\underset{\wedge}{x}_{i+1}$. So $\underset{\wedge}{X}_1; \ldots; \underset{\wedge}{X}_{m+1}$ here is a t_X-determination sequence in which $\underset{\wedge}{X}^o_i = \langle \underset{\wedge}{x}_i \rangle$ for $\underline{i} = 2, \ldots, \underline{m}+1$. (Unless $\langle \underset{\wedge}{x}_1, \ldots, \underset{\wedge}{x}_{m+1} \rangle$ is a total path to $\underset{\wedge}{x}_{m+1}$ in $\underset{\wedge}{X}$, putting also $\underset{\wedge}{X}^o_1 = \langle \underset{\wedge}{x}_1 \rangle$ selects $\underset{\wedge}{x}_1$ as the omission for continuing the precession.) It will be evident from Theorems

3 & 17 that this $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is a standard cd-series. But more than that, since the sequence is clearly $\underset{\wedge}{X}$-wise compact while singleton tuples $\langle \underset{\wedge}{x}_2\rangle,\ldots,\langle \underset{\wedge}{x}_{m+1}\rangle$ are all $\underset{\wedge}{X}$-wise solid and each $\underset{\wedge}{x}_i$ t_X-precedes $\underset{\wedge}{x}_{i+1}$ ($\underline{i} = 1,\ldots,\underline{m}$), $\underset{\wedge}{X}_1;\ldots;\underset{\wedge}{X}_{m+1}$ is an $\underset{\wedge}{X}$-wise chain of t_X-determinations whose special character has been described previously (p. 2.67f.) And the fact that any path to $\underset{\wedge}{x}_{m+1}$ in $\underset{\wedge}{X}$ is the terminal segment of a total path to $\underset{\wedge}{x}_{m+1}$ is just a special case of the molar principle that the precession of stages in an $\underset{\wedge}{X}$-wise chain of t_X-determinations can always be continued until its initial stage $\underset{\wedge}{X}_1$ contains no variable in $\underline{I}(\underset{\wedge}{X})$ that t_X-precedes $\underset{\wedge}{X}_2^o$.

Accordingly, we take our guiding directive for molar path theory to be that a macrocausal path digraph Π_X is above all to represent sequences of omission tuples in chained cd-series, while reducing to a microcausal path digraph in the limiting case wherein all its nodes are singletons. The technicalities in Theorem 20 largely dictate what any such Π_X must be like. First of all, its nodes must be tuples $\{\underset{\wedge}{X}_i\}$ of variables from some base (background) tuple $\underset{\wedge}{X}$. Secondly, Π_X must contain a partial-order relation $\rightarrow\!\!\!\!\circ\!\!\rightarrow$ on Π_X-nodes signifying direct antecedence in Π_X. It will be convenient to call $\rightarrow\!\!\!\!\circ\!\!\rightarrow$ the direct-source relation in Π_X, though we must take care not to confuse this with microcausal direct-source connection in $\underset{\wedge}{X}$ proper. Any node $\underset{\wedge}{X}_i$ on a $\rightarrow\!\!\!\!\circ\!\!\rightarrow$-path to any node $\underset{\wedge}{X}_j$ in Π_X must t_X-precede and be $\underset{\wedge}{X}$-wise causally independent of $\underset{\wedge}{X}_j$; hence in particular $\underset{\wedge}{X}_i$ and $\underset{\wedge}{X}_j$ must be disjoint. The aggregate $\underset{\wedge}{\bar{X}}{}^*_j$ of all nodes directly antecedent to node $\underset{\wedge}{X}_j$ in Π_X must t_X-determine $\underset{\wedge}{X}_j$ while disconnecting all other Π_X-nodes from $\underset{\wedge}{X}_j$. (Here and subsequently, the super-bar in $\bar{X}^*_j$ denotes a subtuple of $\underset{\wedge}{X}$ that is not necessarily in Σ_X.) And last but far from least, interior nodes of Π_X must be $\underset{\wedge}{X}$-wise solid.

Let us say that a set $\Sigma_X = \{\underset{\wedge}{X}_i\}$ of tuples is a <u>partition</u> of tuple $\underset{\wedge}{X}$ just in case (<u>a</u>) each $\underset{\wedge}{X}_i$ in Σ_X is a subtuple of X, and (<u>b</u>) each variable in $\underset{\wedge}{X}$ is in exactly one tuple in Σ_X. (Condition (<u>b</u>) entails that any tuples $\underset{\wedge}{X}_i$ and $\underset{\wedge}{X}_j$ in Σ_X are disjoint unless $\underset{\wedge}{X}_i = \underset{\wedge}{X}_j$, whence in particular $\underset{\wedge}{X}_i \doteq \underset{\wedge}{X}_j$ only if $\underset{\wedge}{X}_i = \underset{\wedge}{X}_j$. And (<u>a</u>)'s requiring each $\underset{\wedge}{X}_i$ and $\underset{\wedge}{X}_j$ in Σ_X not merely to contain only $\underset{\wedge}{X}$-variables but to be subtuples of $\underset{\wedge}{X}$ has the convenient but nonessential consequence that $\underset{\wedge}{X}_j$ contains all variables in

$\mathbf{X}_i$ only if $\mathbf{X}_i$ is a subtuple of $\mathbf{X}_j$.) Then the requirements on Π_X just noted are fulfilled if we stipulate that an (<u>ideal</u>) <u>molar</u> <u>path</u> <u>structure</u> (i.e. macrocausal path digraph) <u>on</u> <u>base</u> $\mathbf{X}$ is any 2-tuple $\Pi_X = \langle \Sigma_X, \rightarrow \rangle$ satisfying the following conditions:

1) Σ_X is a partition $\Sigma_X = \{\mathbf{X}_i\}$ of $\mathbf{X}$ in which every node (tuple) $\mathbf{X}_i$ is $\mathbf{X}$-wise solid; and $\rightarrow$ is a binary relation on Σ_X. (If $X_i \rightarrow X_j$, we say that $\mathbf{X}_i$ is a direct source of $\mathbf{X}_j$ in Π_X and that $\mathbf{X}_j$ is an interior node of Π_X. If $\mathbf{X}_j$ is in Σ_X but has no direct source in Π_X, $\mathbf{X}_j$ is an exterior node of Π_X.)

2) For every node X_j of Π_X, define the Π_X-<u>wise</u> <u>proximal</u> <u>source</u>, $\bar{\mathbf{X}}^*_j$, of $\mathbf{X}_j$ to be the (possibly null) subtuple $\bar{\mathbf{X}}^*_j$ of $\mathbf{X}$ such that each Π_X-node $\mathbf{X}_i$ is either disjoint from or is a subtuple of $\bar{\mathbf{X}}^*_j$ and is the latter iff $\mathbf{X}_i \rightarrow \mathbf{X}_j$. That is, $\bar{\mathbf{X}}^*_j$ comprises just the variables in all direct sources of $\mathbf{X}_j$ in Π_X. Then for each Π_X-node $\mathbf{X}_j$, if $\bar{\mathbf{X}}^*_j$ is non-null, $\bar{\mathbf{X}}^*_j$ t_X-determines $\mathbf{X}_j$ with $\bar{\mathbf{X}}^*_j$ disjoint from $\mathbf{X}_j$ and $\bar{\mathbf{X}}^*_j;\mathbf{X}_j$ $\mathbf{X}$-wise compact (cf. Def. 2.21).

3) Whenever $X_i \rightarrow X_j$ in Π_X, $\mathbf{X}_i$ contains at least one $\mathbf{X}$-wise direct source of some variable in $\mathbf{X}_j$.

4) Each exterior node of Π_X is $\mathbf{X}$-wise causally independent of all other nodes of Π_X.

An immediate consequence of Condition 2 is that $\mathbf{X}_i \rightarrow \mathbf{X}_j$ only if $\mathbf{X}_i$ t_X-precedes $\mathbf{X}_j$ with $\mathbf{X}_i \neq \mathbf{X}_j$; hence from the classical-partial-order status of t_X-precedence and the equivalence of $\doteq$ with $=$ on Σ_X, $\rightarrow$ is a strict (i.e. irreflexive) partial order on Σ_X.

Given any partition Σ_X of $\mathbf{X}$ whose nodes are $\mathbf{X}$-wise solid, Conditions 2-4 provide an explicit definition for $\rightarrow$ on Σ_X that may not, however, satisfy the entirety of Condition 2. Specifically, Conditions 2-4 entail that for any nodes $\mathbf{X}_i$ and $\mathbf{X}_j$ in Σ_X, $\mathbf{X}_i \rightarrow \mathbf{X}_j$ if and only if $\mathbf{X}_i \neq \mathbf{X}_j$ with $\mathbf{X}_i$ containing an $\mathbf{X}$-wise direct source of some variable in $\mathbf{X}_j$. (The <u>only-if</u> part of this is just Condition 3 with irreflexivity added from Condition 2; its <u>if</u> part holds because if $\mathbf{X}_i \neq \mathbf{X}_j$ when $\mathbf{X}_i$ contains an $\mathbf{X}$-wise direct source of some variable in $\mathbf{X}_j$, Condition 4 disallows $\bar{\mathbf{X}}^*_j$ to be null, whence the compactness stipulated in Condition 2 requires $\mathbf{X}_i$ to be included in $\bar{\mathbf{X}}^*_j$.)

Taking this biconditional to define $\overset{\circ}{\longrightarrow}$ gives us that whenever $\underset{\wedge}{\bar{X}}{}^*_j$ is non-null, $\underset{\wedge}{\bar{X}}{}^*_j$ s-determines $\underset{\wedge}{X}_j$ with $\underset{\wedge}{\bar{X}}{}^*_j;\underset{\wedge}{X}_j$ $\underset{\wedge}{X}$-wise compact and $\underset{\wedge}{\bar{X}}{}^*_j$ disjoint from $\underset{\wedge}{X}_j$. It does not, however, insure that $\underset{\wedge}{X}_i$ t_X-precedes $\underset{\wedge}{X}_j$ whenever $\underset{\wedge}{X}_i \overset{\circ}{\longrightarrow} \underset{\wedge}{X}_j$, as needed for $\underset{\wedge}{\bar{X}}{}^*_j$'s s-determination of $\underset{\wedge}{X}_j$ to be t_X-determination as Condition 2 also requires. So what Conditions 2-4 really stipulate, beyond explicit definitions for $\overset{\circ}{\longrightarrow}$ and $\{\underset{\wedge}{\bar{X}}{}^*_j\}$, is that Σ_X so partitions $\underset{\wedge}{X}$ that whenever $\underset{\wedge}{X}_i \neq \underset{\wedge}{X}_j$ therein, $\underset{\wedge}{X}_i$ contains an $\underset{\wedge}{X}$-wise direct source of some variable in $\underset{\wedge}{X}_j$ only if each variable in $\underset{\wedge}{X}_i$ is an $\underset{\wedge}{X}$-wise source (not necessarily a direct one) of some variable in $\underset{\wedge}{X}_j$.

The organization of an ideal molar path structure on $\underset{\wedge}{X}$ is gratifyingly tidy. First of all, each path $\underset{\wedge}{X}_1 \overset{\circ}{\longrightarrow} \underset{\wedge}{X}_2 \overset{\circ}{\longrightarrow} \dots \overset{\circ}{\longrightarrow} \underset{\wedge}{X}_{m+1}$ in Π_X identifies an $\underset{\wedge}{X}$-wise chained t_X-determination sequence $\underset{\wedge}{\bar{X}}_1 \Rightarrow \underset{\wedge}{\bar{X}}_2 \Rightarrow \dots \Rightarrow \underset{\wedge}{\bar{X}}_m \Rightarrow \underset{\wedge}{X}_{m+1}$ wherein $\underset{\wedge}{X}{}^o_{m+1} = \underset{\wedge}{X}_{m+1}$ and precessing from there, for each $\underline{i} = \underline{m},\dots,2$, $\underset{\wedge}{\bar{X}}{}^o_i = \underset{\wedge}{X}_i$ and $\underset{\wedge}{\bar{X}}_{i-1} \doteq \langle \underset{\wedge}{\bar{X}}{}^*_i, \underset{\wedge}{\bar{X}}_i\text{-not-}\underset{\wedge}{\bar{X}}{}^o_i\rangle = \langle \underset{\wedge}{\bar{X}}{}^*_i, \underset{\wedge}{\bar{X}}_i\text{-not-}\underset{\wedge}{X}_i\rangle$. (Proof is immediate from Def. 2.24-1, since $\underset{\wedge}{X}$-wise compactness of all $\underset{\wedge}{\bar{X}}{}^*_i;\underset{\wedge}{X}_i$ entails that $\underset{\wedge}{\bar{X}}_1;\dots;\underset{\wedge}{\bar{X}}_{m+1}$ is X-wise compact, $\underset{\wedge}{X}$-wise solidity of each $\underset{\wedge}{X}_i$ is a basic stipulation, and each omission tuple $\underset{\wedge}{X}_i$ t_X-precedes $\underset{\wedge}{X}_{i+1}$ in the $\overset{\circ}{\longrightarrow}$-path as already noted.) This is <u>exactly</u> like the chained t_X-determination sequences demarked by microcausal digraph paths except for generalizing single-variable omissions to $\underset{\wedge}{X}$-wise solid omission tuples. Also as in the microcausal case, the quasi-causal regularity under which each $\underset{\wedge}{\bar{X}}{}^*_i$ determines $\underset{\wedge}{X}_i$ is pre-emptively embedded in the one under which $\underset{\wedge}{\bar{X}}_{i-1}$ determines $\underset{\wedge}{\bar{X}}_i$. (For the significance of that, see Theorem 24.) That this t_X-determination sequence $\underset{\wedge}{\bar{X}}_1 \Rightarrow \dots \Rightarrow \underset{\wedge}{\bar{X}}_m \Rightarrow \underset{\wedge}{X}_{m+1}$ identified by molar path $\underset{\wedge}{X}_1 \overset{\circ}{\longrightarrow} \dots \overset{\circ}{\longrightarrow} \underset{\wedge}{X}_m \overset{\circ}{\longrightarrow} \underset{\wedge}{X}_{m+1}$ is a cd-series with the pre-emptive embedding just noted is the molar version of causal composition principle <u>CmP</u>-4, and for $\underline{m} = 2$ reduces to the latter when the omission tuples are singletons. Also worth making explicit is that for each interior node $\underset{\wedge}{X}_j$ of Π_X, all microcausal paths within $\underset{\wedge}{X}$ from any variable in $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_j$ to any variable in $\underset{\wedge}{X}_j$ pass through $\underset{\wedge}{\bar{X}}{}^*_j$ (since $\underset{\wedge}{\bar{X}}{}^*_j;\underset{\wedge}{X}_j$ is $\underset{\wedge}{X}$-wise compact), so that $\underset{\wedge}{\bar{X}}{}^*_j$ disconnects $\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_j$ from $\underset{\wedge}{X}_j$. Moreover, from Condition 3, $\underset{\wedge}{\bar{X}}{}^*_j$ is the smallest (least inclusive) aggregate of Π_X-nodes having this disconnection property. That is, for any $\underset{\wedge}{X}$-subtuple $\underset{\wedge}{\bar{X}}{}^+_j$ comprising the variables in some subset of Σ_X not including $\underset{\wedge}{X}_j$, if $\underset{\wedge}{\bar{X}}{}^+_j$ disconnects

$\underset{\wedge}{X}$-not-$\underset{\wedge}{X}_j$ from $\underset{\wedge}{X}_j$ then $\underset{\wedge}{\bar{X}}{}^*_j$ is a subtuple of $\underset{\wedge}{\bar{X}}{}^+_j$. In the limiting case where $\underset{\wedge}{X}_j = \langle \underset{\wedge}{x}_j \rangle$ and all nodes aggregated into $\underset{\wedge}{\bar{X}}{}^*_j$ are singletons, $\underset{\wedge}{\bar{X}}{}^*_j$ is the microcausally proximal source of $\underset{\wedge}{x}_j$ in $\underset{\wedge}{X}$--just as needed if molar path structures are to include microcausal ones as limiting cases.

Secondly, for any two variables $\underset{\wedge}{x}_i$ and $\underset{\wedge}{x}_j$ in distinct Π_X-nodes $\underset{\wedge}{X}_i$ and $\underset{\wedge}{X}_j$, respectively, $\underset{\wedge}{x}_i$ is an $\underset{\wedge}{X}$-wise source of $\underset{\wedge}{x}_j$ only if there is a $-\!\!\circ\!\!\rightarrow$-path in Π_X from $\underset{\wedge}{X}_i$ to $\underset{\wedge}{X}_j$. (Proof: We have already observed in slightly different terms that whenever there is a length-2 path within $\underset{\wedge}{X}$ from a variable in Π_X-node $\underset{\wedge}{X}_h$ to a variable in Π_X-node $\underset{\wedge}{X}_k$, either $\underset{\wedge}{X}_h = \underset{\wedge}{X}_k$ or $\underset{\wedge}{X}_h -\!\!\circ\!\!\rightarrow \underset{\wedge}{X}_k$. From there, completion of the argument is obvious.) Consequently, for any two distinct Π_X-nodes $\underset{\wedge}{X}_i$ and $\underset{\wedge}{X}_j$, $\underset{\wedge}{X}_i$ is $\underset{\wedge}{X}$-wise causally independent of $\underset{\wedge}{X}_j$ just in case there is no $-\!\!\circ\!\!\rightarrow$-path in Π_X from $\underset{\wedge}{X}_i$ to $\underset{\wedge}{X}_j$. And from there, under the partial-order character of $-\!\!\circ\!\!\rightarrow$, it follows that every sequence of nodes in Π_X has at least one permutation under which the sequence is $\underset{\wedge}{X}$-wise well-ordered (cf. Def. 2.22-4)--just as holds for any sequence of single variables in $\underset{\wedge}{X}$. Using this well-ordering principle, for any node $\underset{\wedge}{X}_j$ to which there is a $-\!\!\circ\!\!\rightarrow$-path in Π_X of length 2 or greater, we can construct from the nodes in Π_X an $\underset{\wedge}{X}$-wise solidly conservative t_X-determination sequence (cf. Def. 2.24-2 and Theorem 21) that precesses from $\underset{\wedge}{X}_j$ to the aggregate of exterior Π_X-nodes that t_X-precede $\underset{\wedge}{X}_j$. Specifics on this point need not detain us, however, for they are just an instance of the most basic isomorphism between ideal-macrocausal and microcausal path digraphs.

Most fundamentally, if Π_X is an ideal molar path structure, an exact counterpart of Theorem 1, and hence of all ensuing microcausal theorems, holds for Π_X. Detailing that correspondence would be unnecessarily tedious here. But the point is simply this: If $\underset{\wedge}{X}_m$ is any node of Π_X, either interior or exterior, there is also an ideal molar path structure $\Pi_{X\text{-not-}X_m}$ whose nodes are just the nodes of Π_X excluding $\underset{\wedge}{X}_m$, and whose direct-source connections are derived from those in Π_X exactly as described by Theorem 1 for microcausal direct-source connections in $\underset{\wedge}{X}$ vs. $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$. (Proof will be omitted here, but it follows straightforwardly from the relation just noted between any microcausal path within $\underset{\wedge}{X}$ and the derivative macrocausal path in Π_X.) And whenever $\underset{\wedge}{X}_m$ is an interior node of Π_X, the proximal quasi-causal regularities $\{\underset{\wedge}{X}_i = \phi_i^*(\underset{\wedge}{\bar{X}}_i^*)\}$ in Π_X generate the proximalities in $\Pi_{X\text{-not-}X_m}$ in exact isomorphism to how this occurs microcausally when $\underset{\wedge}{X}$ is reduced to $\underset{\wedge}{X}$-not-$\underset{\wedge}{x}_0$. Specifically, each Π_X-node $\underset{\wedge}{X}_j$ of which $\underset{\wedge}{X}_m$ is not a direct source in Π_X has the same proximal source $\underset{\wedge}{\bar{X}}_j^*$ in $\Pi_{X\text{-not-}X_m}$ as it has in Π_X. But if $\underset{\wedge}{X}_{m-1} \dashrightarrow \underset{\wedge}{X}_m \dashrightarrow \underset{\wedge}{X}_{m+1}$ in Π_X, and we write $\underset{\wedge}{\bar{X}}_m = \underset{\wedge}{\bar{X}}_{m+1}^*$, $\underset{\wedge}{\bar{X}}_{m-1} \doteq \langle \underset{\wedge}{\bar{X}}_m^*, \underset{\wedge}{\bar{X}}_m\text{-not-}\underset{\wedge}{X}_m \rangle$ in accord with our prior observations (p. 2.81) on the compositional import of $\dashrightarrow$-paths, $\underset{\wedge}{\bar{X}}_{m-1} \Rightarrow \underset{\wedge}{\bar{X}}_m \Rightarrow \underset{\wedge}{X}_{m+1}$ is an $\underset{\wedge}{X}$-wise chained t_X-determination sequence of length 2 whose stage $\underset{\wedge}{\bar{X}}_{m-1}$ becomes the proximal source of $\underset{\wedge}{X}_{m+1}$ in reduced molar digraph $\Pi_{X\text{-not-}X_m}$ under the quasi-causal regularity derived by composing into $\underset{\wedge}{X}_{m+1} = \phi_{m+1}^*(\underset{\wedge}{\bar{X}}_{m+1}^*)$ the one under which $\underset{\wedge}{\bar{X}}_{m-1}$ determines $\underset{\wedge}{\bar{X}}_m$ and in which $\underset{\wedge}{X}_m = \phi_m^*(\underset{\wedge}{\bar{X}}_m^*)$ is pre-emptively embedded. In such fashion, the quasi-causal regularity $\underset{\wedge}{X}_k = \phi_{kh}(\underset{\wedge}{\bar{X}}_h)$ under which any given interior node $\underset{\wedge}{X}_k$ of Π_X is determined by an aggregate $\underset{\wedge}{\bar{X}}_h$ of Π_X-nodes not all proximal for $\underset{\wedge}{X}_k$ in Π_X can be derived from Π_X's proximal regularities by iteratively eliminating from Π_X the buffer nodes that are on $\dashrightarrow$-paths between $\underset{\wedge}{\bar{X}}_h$ and $\underset{\wedge}{X}_k$. (Cf. Def. 2.9 and Theorems 4 & 5.) If $\underset{\wedge}{\bar{X}}_h$ t_X-determines $\underset{\wedge}{X}_k$ and the sequence of omission nodes (deleted from right to left) is $\underset{\wedge}{X}$-wise well-ordered, it can easily be seen that $\underset{\wedge}{X}_k = \phi_{kh}(\underset{\wedge}{\bar{X}}_h)$ is the composition of a cd-series (in fact an $\underset{\wedge}{X}$-wise solidly conservative one) whose single-step regularities are, or derive by pre-emptive embedding from, ones that are proximal in Π_X.

The goals set for this chapter have now been essentially achieved. We have studied the logic of causal composability in some depth, and have seen how the complexities of recursive compositions that preserve causality, which are largely intractable in microcausal terms, can be effectively conceptualized as cd-series of quasi-causal molar regularities. And we have observed reasonably general conditions under which, with t-precedence taken as our molar counterpart of the causal-source relation on single variables, the t_X-precedence structure of nodes in a molar partition of base tuple $\underset{\wedge}{X}$ is characterized by principles that are virtually word-for-word translations of the principles that govern microcausal paths in $\underset{\wedge}{X}$. That is quite enough for this occasion. Nevertheless, there is a great deal more to be said about causal macrostructure, and some of what remains for molar digraph theory deserves parting acknowledgment.

First of all, the version of molar path structure defined on p. 2.80 has been labeled "ideal" to recognize that alternatives to Conditions 1-4 may also identify patterns of molar causality that usefully resemble microcausal path structure. What might such alternatives be? Conditions 3 & 4 contribute little to the isomorphism, and can be waived with only minor complications for Π_X's representation of disconnection and $\underset{\wedge}{X}$-wise causal independence. But Conditions 1 & 2 do not easily submit to relaxation. Even so, we do not want disjointness of molar path nodes to be obligatory; for molar attributes that we treat as causally distinct often appear to have overlapping microcausal abstraction bases. There is no evident reason why molar path models cannot admit interlocking nodes, but it will take work.

Then there is the question of how a molar path digraph Π_X on $\underset{\wedge}{X}$ can best be embedded in ones on supertuples of $\underset{\wedge}{X}$. The theory of this should be largely routine, but it still awaits accomplishment.

Above all, given the microcausal path structure π_X within tuple $\underset{\wedge}{X}$, is there any insightful algorithm that can extract from π_X the partitions $\{\Sigma_X\}$ of $\underset{\wedge}{X}$ for which $\Pi_X = \langle \Sigma_X, -\!\!\circ\!\!\rightarrow \rangle$, with $-\!\!\circ\!\!\rightarrow$ suitably defined (cf. p. 2.80), satisfies ideal digraph Conditions 1-4? Let us call such an Π_X a "molar derivative" of π_X. Any π_X has two trivial molar derivatives, the degenerate one having just $\underset{\wedge}{X}$ itself for its only node, and the one in which Σ_X consists of $\underset{\wedge}{X}$'s singleton subtuples. (The latter is not degenerate, but differs from π_X merely in replacing each $\underset{\wedge}{x}$ in $\underset{\wedge}{X}$ by $\langle \underset{\wedge}{x} \rangle$.) But π_X also generally has nontrivial molar derivatives as well. Can these be found by some technique more efficient than generating every partition of $\underset{\wedge}{X}$ for separate appraisal? We have already identified the essential criterion for Σ_X to comprise the nodes in a molar derivative of π_X: Each node $\underset{\wedge}{X}_j$ in Σ_X must be $\underset{\wedge}{X}$-wise solid, and any other node $\hat{X}_i$ ($\neq \hat{X}_j$) that contains an $\hat{X}$-wise direct source of any variable in $\hat{X}_j$ ~~must t_X precede $\hat{X}$.~~ Starting with some π_X already known to be a molar derivative of π_X, is there some way for us to determine with comparative ease that combining certain nodes of π_X into coarser nodes (or, alternatively, splitting certain nodes of π_X) generates another molar derivative π'_X of π_X? What is envisioned here is the following: For any two molar derivatives π_X and π'_X of π_X, say that π'_X is a "coarsening" of π_X iff each node of π_X is a subtuple of some node of π'_X. Then the coarsening relation is a classical partial order--in fact, a lattice with the two trivial cases already noted as extremes--on the set $\underline{MD}(\pi_X)$ of π_X's molar derivatives. And $\underline{MD}(\pi_X)$ is finite, so for each π_X in $\underline{MD}(\pi_X)$, the subset of $\underline{MD}(\pi_X)$ comprising just the immediate successors (alternatively, the immediate predecessors) of π_X in the coarsening order is not only finite but in all likelihood no more than a very small fraction of $\underline{MD}(\pi_X)$. A method for converting any π_X in $\underline{MD}(\pi_X)$ into a list of its immediate successors (or predecessors) then provides orderly identification of all molar derivatives of π_X. Whether insightful procedures of this sort exist and, if they do, just what their value may be for the theory of molar causality, is far from clear. But the abstract question is intrinsically challenging.