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Factor indeterminacy: The saga continues

William W. Rozeboom

Department of Psychology, University of Alberta, Edmonton, Alberta T6G 2E9, Canada

The much-discussed prevailing failure of a moment decomposition $\mathbf{M}_{ZZ} = \mathbf{AM}_0\mathbf{A}'$ to identify just one factor tuple F such that $Z = \mathbf{A}F$ and $\mathbf{M}_{FF} = \mathbf{M}_0$ is only one of many ways in which a selected fragment of a complete factor solution generally specifies the solution's remainder only imperfectly. Precise ranges are worked out here for the main varieties of such indeterminacies, together with the special conditions, if any, under which they shrink to unique determinations.

1. Introduction

In the wake of all the literature, both classical (cf. Steiger, 1979) and modern (McDonald & Mulaik, 1979; Rozeboom, 1982; Williams, 1978), on the much lamented failure of common factors to be generally identifiable from the data variables from which they are inferred, one might well wonder how anything could remain to be said on this matter. Yet my recent work on quadratic factor analysis (Rozeboom & McArdle, forthcoming) has brought home to me that the generic topic of factor indeterminacy is considerably broader than what has been foreground in its extant literature, and that although comprehensive study of this has little direct bearing on multivariate practice, fragments of its returns are relevant to the theory of quad-factoring and, I should anticipate, other complexely structured models that may be forthcoming.

There are, in fact, three groups of factor-indeterminacy issues: Mathematical, Epistemological, and Motivational, the last comprising efforts to say how the others matter. Present concern is mainly with the first of these; but the first two have become so obscurely fused that I must begin by prying them apart. Specifically, without arguing the case in detail, I shall submit that much past distress over factor indeterminacy has been an implicit desire for factors to be *identified* in an epistemic sense much stronger than unique specification, a sense that we don't know how to cash out even for data variables. Once freed of this beguilement, we can focus on the modest mathematical points of model specification that occasion this paper.

2. The nature of model 'indeterminacy'

Precisely what is to be meant by describing a multivariate model as 'identified' or, contrastingly, as 'indeterminate' in some particular application is surprisingly problematic. The indeterminacies of present concern are in the generic linear factor model which, applied to analysis of a sample distribution of scores on an n-tuple $Z = \langle z_1, \ldots, z_n \rangle$ of data variables, hypothesizes that the observed second-order Z-moment matrix $\mathbf{M}_{ZZ}$ has a decomposition of form†

$$\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_{FF}\mathbf{A}', \tag{1a}$$

wherein $\mathbf{M}_{FF}$ is the second-order moment matrix in this sample for some m-tuple $F = \langle f_1, \ldots, f_m \rangle$ of factors that generate the Z-data according to structural equations

$$Z = \mathbf{A}F. \tag{1b}$$

Given a particular $\mathbf{M}_{ZZ}$, before (1) can be solved even for $\mathbf{A}$ and $\mathbf{M}_{FF}$ much less F additional constraints are needed. In the common-factor species of (1), pattern matrix $\mathbf{A}$, factor tuple F, and F-moment matrix $\mathbf{M}_{FF}$ are required to have structure

$$\mathbf{A} = [\mathbf{A}_1\ \mathbf{I}], \qquad F = \langle F_1, U \rangle = \langle f_1, \ldots, f_r, u_1, \ldots, u_n \rangle,$$
$$\mathbf{M}_{FF} = \begin{bmatrix} \mathbf{M}_1 & 0 \\ 0 & \mathbf{D}^2 \end{bmatrix},$$

with $\mathbf{A}$ of order $n \times r$ for $r < n$ and $\mathbf{D}$ $n \times n$ diagonal; whence the model can also be written as

$$\mathbf{M}_{ZZ} = \mathbf{A}_1\mathbf{M}_1\mathbf{A}_1' + \mathbf{D}^2, \quad \mathbf{M}_{F_1F_1} = \mathbf{M}_1, \quad \mathbf{M}_{UU} = \mathbf{D}^2, \quad \mathbf{M}_{F_1U} = \mathbf{0} \tag{2a}$$

$$Z = \mathbf{A}F_1 + U. \tag{2b}$$

(Alternatively, of course, we can equivalently stipulate $\mathbf{M}_{UU} = \mathbf{I}$ with $\mathbf{D}$ the pattern on U.) Without still more constraints, however, even restricted model (2) remains indeterminate in that for a given Z with given moments $\mathbf{M}_{ZZ}$, there are in general many different alternatives for $\langle r, \mathbf{A}_1, \mathbf{M}_1, \mathbf{D}, F_1, U \rangle$ in (2) that satisfy the model equations if any does; and the task of model specification is to reduce this range of model solutions by stipulation of side conditions. In the limit, increasing the latter may yield a fully determinate model whose solution-set is a singleton. But that is only a theoretical ideal never strictly attained nor often even closely approximated in practice. Some of the obstacles to strict model identification verge upon triviality, such as that our solution for $\langle \mathbf{A}, \mathbf{M}_{FF} \rangle$ always suffers from rounding error and (what is not quite so trivial) that the $\mathbf{M}_{ZZ}$ exactly reproduced by model fit in (1a) or (2a) comprises not literally data moments but at best imperfect approximations thereto found by minimizing a loss-function chosen more for mathematical convenience than because we believe it to be interpretively optimal. But a far more serious problem for model identification is that, in a tough epistemological sense, we never know precisely what we are talking about when we fit models to data.

Roughly speaking, we may say that model (1) is (fully) 'determinate' in some particular application with side constraints just in case its totality of imposed

† I write $\mathbf{M}$ for second-order moment matrices rather than $\mathbf{C}$ or $\mathbf{\Sigma}$ for traditional covariances because present results apply equally to centred and uncentred moments, and some modern models (quadratic factor analysis in particular) are best formulated in terms of uncentred variables with the additive constants in linear dependencies treated as coefficients on a factor constant at unity. But little will be lost here if you take $\mathbf{M}$ to comprise the centred covariances between whatever tuples of variables are denoted by $\mathbf{M}$'s subscripts.

conditions provides identification of exactly one model solution. But the notion of 'identifying' something, model solutions in particular, is obscure. In first approximation, to identify an entity s is to communicate a name, description, or other denotative phrase that picks out this particular s as differentiated from all other things we regard as distinct from s. However, not all expressions that refer to the same s are equally acceptable as *identifications* thereof. To give quantitative examples, the description 'Mean number of acorns collected per squirrel in Ohio last October' designates a specific number while leaving us egregiously ignorant of its identity. And if a math student is instructed to find the largest root of equation $x^2 - 5x + 6 = 0$, the answer 'Three' is correct; but 'The smallest root of this equation plus one' would be accepted only if the student can go on to say *what* number that is, and 'The largest number whose product with five less its square equals six' would be viewed as abject failure to *identify* the solution even though this description does indeed refer to it. Identification requires not merely individuating reference, but reference in whatever special way we intuitively require for greatest epistemic illumination.

The point is this: On pain of dismissing the past factor-indeterminacy literature as foolish, we surely do not want to say that side conditions on (1) make the model 'determinate' whenever they specify a unique solution. For we can always supplement our mathematical constraints by

> Moreover, $\langle \mathbf{A}, \mathbf{M}_{FF}, F \rangle$ is the particular solution of (1) that most closely aligns F with an m-tuple of Z's causal sources. (3)

(Degree of 'alignment' here can be made precise as, say, the F-axes' mean correlation with the source variables to which they are respectively matched.) Indeed, something like (3) is already an implicit presumption in most applied multivariate research. Although our understanding of causality is still primordial (cf. my unpublished 'Mentality and the Deeper Logic of Lawfulness'), there can be little doubt that any tuple Z of data variables does in fact have causal sources which, moreover, comprise just a vanishingly small subset of the variables with which Z is jointly distributed. So we have every reason to presume that in most applications of (1) to data on a determinate Y, inclusion of a suitably precise version of (3) among our model constraints indeed specifies a unique solution. Yet that does little to allay traditional *angst* over factor indeterminacy. For even if descriptor

> The $\langle \mathbf{A}, \mathbf{M}_{FF}, F \rangle$ that satisfies (1) while having further properties [such-and-such], and for which (3) also holds

picks out just one factor tuple F, it neither *identifies* that F nor gives any clue to how its identity might be found.

To identify any particular solution of (1), we must designate its $\langle \mathbf{A}, \mathbf{M}_{FF}, F \rangle$ by expressions of whatever canonical forms we have judged to be most useful for dealing with entities of these kinds. Happily, coefficient and moment matrices present no puzzles in this regard, insomuch as intuition insists that the canonical form for identifying a finite array of numbers is listing for each element thereof a symbol in standard numeric notation which designates that number. (Cf. Whereas 'The square-root of 1.96' specifies $\sqrt{1.96}$ without identifying it, '1.4' does both.) *But we have no* *canonical forms of expression for identifying variables, nor any theory of what should go into one.* In fact, it is difficult to find paradigm descriptions of variables in research practice that specify their intended referents well, let alone identify them.

The claim I have just put is too contentious for easy probate. So I invite you to test it yourself by contemplating how, when preparing an empirical research report, you would attempt to identify your study's data variables. Simply publishing your observed score matrix would accomplish little, for that tells nothing about *what* the variables are on which those numbers are scale values. More informative is for you to describe the procedures that elicited this output from your sample subjects in a way that defines how scores on these very same variables are to be obtained for other subjects in whatever population your study is construed to sample. Yet however exhaustively you spell out your procedures – and in practice we seldom manage to say much – it will always be possible to detail them further in conflicting ways (e.g. different constraints on diurnal time of observation, on intensities of ambient heat and light, on character of background or even foreground stimuli, etc.) that all fit your particular sample but make some difference for the scores of other subjects really or hypothetically so observed and hence define somewhat different empirical variables. Not merely are our descriptions of data variables always imprecise, we don't even have much notion of what most saliently belongs in such a description in contrast to what should be left out. (If you had unlimited time and patience, how would you decide when you had said enough? And do procedures alone suffice to specify data variables, or does their individuation require other sorts of information as well?)

Our failure either to articulate a reasoned methodology for identifying variables – *any* variables – or to establish some praxis of doing this effectively has seriously impeded psychology's development as a hard science (cf. my 'Mentality and the Deeper Logic of Lawfulness'), and is undoubtedly the most important of factor indeterminacy's neglected facets. It is not, however, my present concern. Rather, once it is plain that epistemologically identified solutions of model (1) are an unattainable ideal, if only because we never have fully determinate conceptions even of the data variables to which we apply this, we are free to explore the mathematics of how $\mathbf{M}_{ZZ}$ and Z conjoin effective side conditions on (1) to limit its solution alternatives without concern for the quality of our knowledge of $\mathbf{M}_{ZZ}$ and Z. I stress this point, because the primary applications envisioned for the theorems below are cases wherein Z comprises not data variables in the most brutely empirical sense but their common or true parts, i.e. with the diagonal of $\mathbf{M}_{ZZ}$ containing estimates of communalities or reliabilities. (That is, $\langle \mathbf{A}, \mathbf{M}_{FF}, F \rangle$ will paradigmatically be the $\langle \mathbf{A}_1, \mathbf{M}_1, F_1 \rangle$ part of common-factor model (2).) The epistemic indeterminacies of data variables' common-parts or true-parts differs only in degree, not in kind, from that of the data variables whose salient components they are.

3. What variables are – sort of

As you will see, the primary factor indeterminacies at issue here concern identifiable solution alternatives just in submodel (1a) for moments. But alternatives for F are also part of the story, so we need some technical standard of individuality for variables. Ontologically, a 'variable' over a population P is a contrast-class of properties (attributes, features, characteristics) that are mutually exclusive and jointly exhaustive over P – i.e. any individual that satisfies the conditions for belonging to P necessarily has one and only one property in this class. (Our conceptual difficulties in individuating properties is why we can so seldom specify variables with much precision.) But when a variable is numerically scaled, as we presume here for all variables at issue, it defines a function mapping each member of its domain P into a number that represents on this scale that individual's particular property in this contrast-class. Accordingly, we shall stipulate that mathematically, in a sense that philosophers characterize as 'extensional', a (numerically scaled, extensional) variable over population P simply *is* a function that maps each member of its domain into one particular number. Then if x and y are both variables over P, they are moreover the *same* variable just in case they are identical as functions, i.e. iff they have the same value for the same argument everywhere in P.

This extensional criterion for the individuation of variables has important deficiencies. One is its disallowing the possibility (a more realistic one than you may at first appreciate) that two ontologically distinct variables may be in perfect one-to-one correlation over P. And it provides no meaningful way to distinguish probabilities from relative frequencies in P unless the number of extant P-members (past, present, and future) is literally infinite. In particular, it does not allow us to entertain hypotheses about distributions of variables under population-defining conditions that happen never to be satisfied. Even so, this criterion does individuate variables up to extensional equivalence; and that seems good enough for the mathematics of factor determinacy. Indeed, it enables us in principle – never mind feasibility in practice – to *identify* extensional variables Z over a finite population P by numerically listing the Z-defining score matrix in P. And given attainable knowledge (or suppositions) κ about variables Z and F, notably a solution for all or part of $\langle \mathbf{A}, \mathbf{M}_{FF} \rangle$ in model (1), we can say that F is (extensionally) 'identifiable' from Z given κ whenever, from any numerically identified value $\mathbf{z}$ of Z, we can effectively compute (up to rounding error) a numerically identified score vector $\mathbf{f}$ such that if κ is true, $\mathbf{f}$ is the one and only vector of scores on F compatible for a member of P with score-vector $\mathbf{z}$ on Z. This is a *relative* indentifiability of F from Z given κ indifferent to whether we ever in fact numerically identify the values of Z for any P-members.

4. Varieties of factor indeterminacy

Given the second-order moment matrix $\mathbf{M}_{ZZ}$ (in P) for variables $Z = \langle z_1, \ldots, z_n \rangle$, a complete order-$m$ solution of model (1) consists of an $n \times m$ real matrix $\mathbf{A}_i$, and an m-tuple $F = \langle f_1, \ldots, f_m \rangle$ of (extensional) variables such that

$$\mathbf{M}_{ZZ} = \mathbf{A}_i \mathbf{M}_i \mathbf{A}_i', \qquad Z = \mathbf{A}_i F_i, \qquad \mathbf{M}_{F_iF_i} = \mathbf{M}_i .$$

Then the generic issue of factor determinacy is the extent to which, under what circumstances, some distinguished fragment of a complete solution specifes the solution's remainder. If we say that a 'main' solution-fragment is any that either contains all of $\mathbf{A}_i$, or of $\mathbf{M}_i$, or of F_i, or none of it, a complete solution has six main fragments, partitioning the generic determinacy issue into six main varieties:

> *Factor-determinacy Question* **A**. Given pattern **A**, what is the range of solutions for $\mathbf{M}_i$ and F_i in $\langle \mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_i\mathbf{A}', Z = \mathbf{A}F_i, \mathbf{M}_{F_iF_i} = \mathbf{M}_i \rangle$? In particular, when are these solution-sets singletons?
>
> *Factor-determinacy Question* **M**. Given moments $\mathbf{M}_0$, what is the range of solutions for $\mathbf{A}_i$ and F_i in $\langle \mathbf{M}_{ZZ} = \mathbf{A}_i\mathbf{M}_0\mathbf{A}_i', Z = \mathbf{A}_iF_i, \mathbf{M}_{F_iF_i} = \mathbf{M}_0 \rangle$?
>
> *Factor-determinacy Question* **F**. Given variables F, what is the range of solutions for $\mathbf{A}_i$ and $\mathbf{M}_i$ in $\langle \mathbf{M}_{ZZ} = \mathbf{A}_i\mathbf{M}_i\mathbf{A}_i', Z = \mathbf{A}_iF, \mathbf{M}_{FF} = \mathbf{M}_i \rangle$?
>
> *Factor-determinacy Question* **AM**. Given pattern **A** and moments $\mathbf{M}_0$, what is the range of solutions for F_i in $\langle \mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_0\mathbf{A}', Z = \mathbf{A}F_i, \mathbf{M}_{F_iF_i} = \mathbf{M}_0 \rangle$? In particular, when is this solution-set a singleton?
>
> *Factor-determinacy Question* **AF**. Given pattern **A** and variables F, what is the range of solutions for $\mathbf{M}_i$ in $\langle \mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_i\mathbf{A}', Z = \mathbf{A}F, \mathbf{M}_{FF} = \mathbf{M}_i \rangle$?
>
> *Factor-determinacy Question* **MF**. Given variables F with moments $\mathbf{M}_0$, what is the range of solutions for $\mathbf{A}_i$ in $\langle \mathbf{M}_{ZZ} = \mathbf{A}_i\mathbf{M}_0\mathbf{A}_i', Z = \mathbf{A}_iF, \mathbf{M}_{FF} = \mathbf{M}_0 \rangle$?

However, because F uniquely specifies $\mathbf{M}_{FF}$ (in P), Question **AF** is trivial while **MF** is equivalent to **F**. The four cases that remain, namely, **A**, **AM**, **F**, and **M** are herewith studied by Theorems 2–5 after Theorem 1 sets out the principle that dominates these Questions.

More specifically, the issues addressed by Theorems 1–5 are respectively

(1) For a given tuple Z of variables with moments $\mathbf{M}_{ZZ}$, and a distinguished pattern matrix **A** such that $\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_0\mathbf{A}'$ for some $\mathbf{M}_0$, under what circumstances is the solution for $\mathbf{M}_i$ in $\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_i\mathbf{A}'$ unique? And when the $\mathbf{M}_i$ therein is indeed unique (at $\mathbf{M}_0$) for this Z, does that also suffice to specify F?

> *Comment*. Unlike the epistemic elusiveness of factor identities, any solution $\langle \mathbf{A}_i, \mathbf{M}_i \rangle$ of $\mathbf{M}_{ZZ} = \mathbf{A}_i\mathbf{M}_i\mathbf{A}_i'$ is in principle numerically identifiable by us up to rounding error. If we were to scan the set of all these solution alternatives and pick one, $\langle \mathbf{A}, \mathbf{M}_0 \rangle$, whose pattern seems closest to what we want at this stage of the analysis, when does choosing **A** for $\mathbf{A}_i$ restrict our choice of $\mathbf{M}_i$ just to $\mathbf{M}_0$? And if this **A** so fixes $\mathbf{M}_i$, does it fix F_i in $Z = \mathbf{A}F_i$ as well? Any **A** having full column rank, a property I shall call 'L(eft)-invertibility', does indeed close out (1)'s solution alternatives in this way.

(2) When $\langle \mathbf{A}, \mathbf{M}_0 \rangle$ is an identified solution of $\mathbf{M}_{ZZ} = \mathbf{A}_i\mathbf{M}_i\mathbf{A}_i'$ with the particular **A** therein distinguished for the purpose at hand but not L-invertible, what is the range of moment matrices $\mathbf{M}_i$ and factor tuples F_i such that $\langle \mathbf{A}, \mathbf{M}_i \rangle$ is a solution of $\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_i\mathbf{A}'$, and F_i a solution of $Z = \mathbf{A}F_i$, for this distinguished **A**?

> *Comment*. The practical import of this question is simply that if $\mathbf{M}_{ZZ}$ and **A** do not suffice to specify $\mathbf{M}_i$ in $\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_i\mathbf{A}'$ when **A** is the pattern we want, we need to decide *which* $\mathbf{M}_i$ is our preferred companion for **A** in subsequent stages of the analysis. Identifying the range of $\mathbf{M}_i$-alternatives may or may not help us to

conceptualize and solve for the one we favour; but it is technical information about this situation that belongs on record. And identifying the range of alternatives for F_i in $Z = \mathbf{A}F_i$ helps us to understand the mathematical nature of classic factor indeterminacy even though it has little evident practical utility.

(3) When $\langle \mathbf{A}, \mathbf{M}_0 \rangle$ is an identified solution of $\mathbf{M}_{ZZ} = \mathbf{A}_i\mathbf{M}_i\mathbf{A}_i'$ with $\mathbf{A}$ and $\mathbf{M}_0$ both distinguished for the purpose at hand but $\mathbf{A}$ not L-invertible, what is the range of factor tuples F_i that are joint solutions of $Z = \mathbf{A}F_i$ and $\mathbf{M}_{F_iF_i} = \mathbf{M}_0$?

Comment. This is the classic factor-indeterminacy question that confronts us when F_i includes both common and unique factors. But in principle, it can also arise when $\mathbf{A}$ is a pattern just on common factors. Although Guttman (1955) has already dealt incisively with this case, Theorem 3 greatly enhances the perspicuity of its geometry.

(4) When $Z = \mathbf{A}F$ for some distinguished factor tuple F whose moment matrix $\mathbf{M}_{FF}$ is singular, what is the range of alternative patterns $\mathbf{A}_i$ such that $Z = \mathbf{A}_iF$ for this same F?

Comment. Rozeboom & McArdle, forthcoming, demonstrate that some structural models may well have reason to seek factor axes that contain linear dependencies. For such solutions, the factor pattern is not unique even when the factors are fixed, and we need to decide which pattern alternatives on our favoured F best serve our interests. The most salient finding in Theorem 4 is that if factors F lie in Z-space, the set of patterns satisfying $Z = \mathbf{A}_iF$ always includes some that are L-invertible – which tells us that we are free to impose L-invertibility of common-factor pattern as a model constraint even when we allow dependencies among the factors. Beyond that, Theorem 4's pattern-range specification may not help much for pattern selection in practice; but we get it for free with proof of the salient point and it has a certain mathematical charm worth savouring.

(5) When Z-moment decomposition $\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_0\mathbf{A}'$ distinguishes $\mathbf{M}_0$, what are the ranges of $\mathbf{A}_i$ and F_j such that $\mathbf{M}_{ZZ} = \mathbf{A}_i\mathbf{M}_0\mathbf{A}_i'$ and $Z = \mathbf{A}_iZ_j$ with $\mathbf{M}_{Z_jZ_j} = \mathbf{M}_0$?

Comment. This case's indeterminacies are too exceptionlessly extensive to suggest much practical use for its findings beyond its restriction $\mathbf{M}_0 = \mathbf{I}$ long familiar to and heavily exploited by traditional factor analysis. But we include it for completeness, especially since it is little more than a corollary of Theorem 4.

Certain technical concepts and matrix principles to be used here need some outset clarification. I have already declared that $\mathbf{M}_{ZZ}$, $\mathbf{M}_{FF}$, etc., are to be matrices of uncentred second-order moments for scales with arbitrary origins (also arbitrary variances). Beyond that, with apologies for belabouring basics, you should be apprised:

(1) All 'variables' cited are presumed to have a joint distribution in some given population P to which all moments and dependencies among these variables are relative. Each tuple of variables $Z = \langle z_1, \ldots, z_n \rangle$, $F = \langle f_1, \ldots, f_m \rangle$, etc., is to be construed extensionally either as a Variables-by-Subjects matrix of real scores in a finite P or, alternatively, as a column vector of real-valued functions over a

hypothetical infinite P for which a joint probability distribution is somehow defined. Under the first reading, $\mathbf{M}_{ZZ} = p^{-1}\mathbf{ZZ}'$ for the p-columned matrix $\mathbf{Z}$ of scores on Z in P; under the second, the ijth element of $\mathbf{M}_{ZZ}$ is $\exp[z_i z_j]$ in P.

(2) Two variables x and y are 'orthogonal' (to each other) in P iff either $\mathbf{xy}' = 0$ where $\mathbf{x}$ and $\mathbf{y}$ are the row vectors of scores on x and y in a finite P, or $\exp[xy] = 0$ in the infinite-population case. This contrasts with the usual definition of orthogonality between variables as zero covariance, though for centred variables the difference vanishes.

(3) As usual, a 'space' of variables (in P) is a set of variables that is closed (in P) under homogeneous linear combinations of its members – 'homogeneity' meaning no additive constants except as coefficients on the unit variable. Likewise as usual, the (one-and-only) space 'spanned' by a tuple Y of variables comprises all variables that are homogeneous linear combinations of the Y-variables. So an n-tuple Z of variables lies in the space spanned by an m-tuple F of variables just in case $Z = \mathbf{A}F$ for some $n \times m$ coefficient matrix $\mathbf{A}$. Y is a 'basis' for the space it spans just in case no proper subtuple of Y also spans Y-space.

(4) For any n-tuple Z and m-tuple F of variables, and any particular $n \times m$ coefficient matrix $\mathbf{A}$, we shall say that F '$\mathbf{A}$-factors' Z iff $Z = \mathbf{A}F$. Evidently, F $\mathbf{A}$-factors Z for at least one $\mathbf{A}$ just in case all Z-variables lie in F-space.

(5) To avoid certain ambiguities in extant terminology, we shall say that a matrix $\mathbf{R}$ is *rectinormal* iff it is orthonormal by columns, i.e. iff $\mathbf{R}'\mathbf{R} = \mathbf{I}$, and *orthonormal* iff both it and its transpose are rectinormal, i.e. iff $\mathbf{R}'\mathbf{R} = \mathbf{RR}' = \mathbf{I}$. A rectinormal matrix is orthonormal just in case it is square; otherwise, it is vertically rectangular.

(6) Any matrix designated by some subscripted $\mathbf{R}$ or $\mathbf{S}$ is stipulated to be rectinormal. Any designated by a subscripted $\mathbf{D}$ is positive diagonal, i.e. with all roots greater than zero. And any designated by a subscripted $\mathbf{M}$ is Gramian but not necessarily non-singular.

(7) A 'generalized' inverse of any $n \times m$ matrix $\mathbf{A}$ is any $m \times n$ matrix $\mathbf{A}^G$ such that $\mathbf{AA}^G\mathbf{A} = \mathbf{A}$. The 'pseudo-inverse', $\mathbf{A}^+$, of $\mathbf{A} \neq \mathbf{0}$ is the special generalized inverse of $\mathbf{A}$ such that $\mathbf{A}^+ = \mathbf{R}_2\mathbf{D}^{-1}\mathbf{R}_1'$ for any basic-structure decomposition $\mathbf{A} = \mathbf{R}_1\mathbf{D}\mathbf{R}_2'$ of $\mathbf{A}$, i.e. where $\mathbf{R}_1$ and $\mathbf{R}_2$ are rectinormal and $\mathbf{D}$ is positive diagonal. (It is well known that $\mathbf{A}$ can always be so decomposed, with $\mathbf{R}_1 = \mathbf{R}_2$ if $\mathbf{A}$ is Gramian, and that the column-order $\mathbf{R}_1$, $\mathbf{D}$ and $\mathbf{R}_2$ is the rank of $\mathbf{A}$.) It can be shown that $\mathbf{A}^+$ not only exists but is unique for any $\mathbf{A} \neq \mathbf{0}$. $\mathbf{A}$ has full column rank, i.e. rank equal to its column-order, just in case matrix $\mathbf{R}_2$ in its basic-structure decomposition $\mathbf{A} = \mathbf{R}_1\mathbf{D}\mathbf{R}_2'$ is square and hence orthonormal. In this important special case, $\mathbf{A}^+$ is also a *left-inverse* of $\mathbf{A}$ in that $\mathbf{A}^+\mathbf{A} = \mathbf{R}_2\mathbf{D}^{-1}\mathbf{R}_1'\mathbf{R}_1\mathbf{D}\mathbf{R}_2' = \mathbf{R}_2\mathbf{R}_2' = \mathbf{I}$. We shall occasionally write $\mathbf{A}^L$ for $\mathbf{A}^+$ when this is a left-inverse of $\mathbf{A}$, and will say that $\mathbf{A}$ is L(eft)-invertible iff $\mathbf{A}^L$ exists. If $\mathbf{A}$ is L-invertible, $\mathbf{A}^L = \mathbf{A}^+ = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'$. When $\mathbf{A}^L$ does *not* exist, we shall say that $\mathbf{A}$ is *L(eft)-ambiguous*, while the *degree* of $\mathbf{A}$'s L-ambiguity is its column-order minus its rank. (*Note*. $\mathbf{A}^+$ is not the only generalized inverse $\mathbf{A}^G$ of $\mathbf{A}$ unless $\mathbf{A}$ is non-singular square. And if $\mathbf{A}^L$ exists, every other $\mathbf{A}^G$ is also a left-inverse of $\mathbf{A}$. But $\mathbf{A}^+$ has special virtues and is the only $\mathbf{A}^G$ we shall exploit here.)

(8) For any matrix $\mathbf{A} \neq \mathbf{0}$ of column-order m, $\mathbf{P}_A =_{\text{def}} \mathbf{A}^+\mathbf{A}$ is the 'projector' into $\mathbf{A}$-row space, while $\mathbf{Q}_A =_{\text{def}} \mathbf{I} - \mathbf{P}_A$ is the (projective) 'complement' of $\mathbf{P}_A$. We use variously subscripted $\mathbf{P}$ and $\mathbf{Q}$ exclusively for row-space projectors and their

complements. (Any $\mathbf{A}$ also has a column-space projector $\mathbf{AA}^+$; but that will not be needed here.) From any basic-structure decomposition of $\mathbf{A}$ as $\mathbf{A} = \mathbf{R}_1\mathbf{DR}_2'$, it is easily seen: (*a*) $\mathbf{P}_A = \mathbf{R}_A\mathbf{R}_A'$ is a (non-unique) basic-structure decomposition of $\mathbf{P}_A$, and moreover, for any basic-structure decomposition $\mathbf{Q}_A = \mathbf{S}_A\mathbf{S}_A'$ of $\mathbf{P}_A$'s complement, $[\mathbf{R}_A\mathbf{S}_A]$ is $m \times m$ orthonormal. (Occasionally, I will refer to any $\mathbf{Q}_A$ so related to $\mathbf{R}_A$ as an 'orthonormal completion' of $\mathbf{R}_A$.) (*b*) $\mathbf{P}_A$ and $\mathbf{Q}_A$ are symmetric and idempotent, i.e. $\mathbf{P}_A' = \mathbf{P}_A = \mathbf{P}_A^2$ and similarly for $\mathbf{Q}_A$, with $\mathbf{P}_A\mathbf{Q}_A = \mathbf{0}$. (*c*) for any order-$m$ row vector $\mathbf{x}$, $\mathbf{xP}_A = \mathbf{x}$ and $\mathbf{xQ}_A = \mathbf{0}$ iff $\mathbf{x}$ lies in the vector space spanned by the rows of $\mathbf{A}$, whereas $\mathbf{xP}_A = \mathbf{0}$ and $\mathbf{xQ}_A = \mathbf{x}$ iff $\mathbf{x}$ is orthogonal to the rows of $\mathbf{A}$. More generally, $\mathbf{x} = \mathbf{x}(\mathbf{P}_A + \mathbf{Q}_A) = \mathbf{xP}_A + \mathbf{xQ}_A$ is a decomposition of $\mathbf{x}$ into a component $\mathbf{xP}_A$ lying in $\mathbf{A}$-row space plus a residual $\mathbf{xQ}_A$ orthogonal to $\mathbf{A}$-row space. (*d*) If r is the rank of $\mathbf{P}_A$, the rank of $\mathbf{Q}_A$ is $m-r$ (including limiting case $\mathbf{Q}_A = \mathbf{0}$ when $r = m$).

(9) Guttman (1955, Lemma 1) has shown, conditional on a minor requirement which he does not make explicit, that whenever the second-order moment matrix $\mathbf{M}_{ZZ}$ for variables Z in P has a decomposition $\mathbf{M}_{ZZ} = \mathbf{AM}_0\mathbf{A}'$, there exists a tuple of (extensional) variables F over P that $\mathbf{A}$-factors Z while $\mathbf{M}_{FF} = \mathbf{M}_0$. I shall refer to this finding as 'Guttman's Lemma'. Its implicit requirement is that the cardinality of P must not be less than the rank of $\mathbf{M}_0$.

The theorems that follow are primarily though not exclusively motivated by multivariate structures wherein the number m of factors in $Z = \mathbf{A}F$ is no greater than the number n of variables factored. This condition prevails when Z is what remains of data variables from which unique factors and sometimes other components have been removed. That we do not literally know Z as an identified score matrix in such cases has no bearing on the *relative* identifications at issue here, i.e. the degree to which scores on F *could* be computed for members of P were we to know their scores on Z.

Theorem 1. Let $Z = \langle z_1, \ldots, z_n \rangle$ be an n-tuple of variables jointly distributed in some population P with second-order moment matrix $\mathbf{M}_{ZZ}$. And suppose that for some integer m, $\mathbf{M}_{ZZ}$ has decomposition

$$\mathbf{M}_{ZZ} = \mathbf{AM}_0\mathbf{A}'$$

for some $n \times m$ matrix $\mathbf{A}$ and $m \times m$ Gramian matrix $\mathbf{M}_0$ that may be singular. Then if factors F_+ are derived from variables Z according to

$$F_+ =_{\text{def}} \mathbf{A}^+ Z,$$

we have: (*a*) F_+ is a tuple of variables in Z-space that $\mathbf{A}$-factors Z, i.e. $Z = \mathbf{A}F_+$. (*b*) If $\mathbf{A}^+$ is moreover a left-inverse of $\mathbf{A}$, i.e. if $\mathbf{A}$ is L-invertible, then $\mathbf{M}_{F_+F_+} = \mathbf{M}_0$ while F_+ is the only tuple of variables that $\mathbf{A}$-factors Z.

Proof. Let $G = \langle g_1, \ldots, g_r \rangle$ comprise the positive-root principal components of the Z-distribution in P. (Evidently $r \leqslant n$ and $r \leqslant m$. Any other basis for Z-space would serve here almost as well as G.) Then $Z = \mathbf{R}_g G$ for some $n \times r$ rectinormal $\mathbf{R}_g$, while G's moment matrix $\mathbf{M}_{GG}$ is non-singular. And since $\mathbf{M}_{ZZ} = \mathbf{R}_g\mathbf{M}_{GG}\mathbf{R}_g'$, our stipulated $\mathbf{M}_{ZZ} = \mathbf{AM}_0\mathbf{A}'$ entails $\mathbf{R}_g\mathbf{M}_{GG}\mathbf{R}_g' = \mathbf{AM}_0\mathbf{A}'$, whose post-multiplication by $\mathbf{R}_g\mathbf{M}_{GG}^{-1}$ yields

$$\mathbf{R}_g = \mathbf{AW}_g \qquad (\mathbf{W}_g =_{\text{def}} \mathbf{M}_0\mathbf{A}'\mathbf{R}_g\mathbf{M}_{GG}^{-1}).$$

Consequently

$$Z = \mathbf{AW}_g G,$$

insertion of which into F_+'s definition gives $F_+ = \mathbf{A}^+\mathbf{AW}_g G$ and hence

$$\mathbf{A}F_+ = \mathbf{AA}^+\mathbf{AW}_g G = \mathbf{AW}_g G = Z$$

as claimed in (*a*). As for (*b*), if $\mathbf{A}^+$ is a left-inverse of $\mathbf{A}$, $\mathbf{M}_{F_+F_+} = \mathbf{A}^+\mathbf{M}_{ZZ}\mathbf{A}^{+\prime} = \mathbf{A}^L(\mathbf{AM}_0\mathbf{A}')\mathbf{A}^{L\prime} = (\mathbf{A}^L\mathbf{A})\mathbf{M}_0(\mathbf{A}^L\mathbf{A})' = \mathbf{M}_0$; while for any F such that $Z = \mathbf{A}F$, if $\mathbf{A}^L$ exists then $F = \mathbf{AL}^L\mathbf{A}F = \mathbf{A}^+Z = F_+$. ∎

Theorem 1 accomplishes three things. First, it shows for any $\mathbf{A}$ that if Z has any $\mathbf{A}$-factors at all it has a special one, F_+, which lies in Z-space and is identifiable just from $\mathbf{A}$ and Z without need for information about the factor moments. Secondly, Theorem 1 largely answers the second part of Factor Determinacy Questions **A** and **AM**, though it leaves open whether the **A** and **AM** solution-sets are ever singletons even when $\mathbf{A}$ is L-ambiguous. (We shall see that this is indeed possible for case **AM**.) And thirdly, the theorem makes clear that so long as $\mathbf{A}$ is L-invertible, the unique existence and identifiability from $\langle \mathbf{A}, Z\rangle$ of an F that $\mathbf{A}$-factors Z is in no way compromised by linear dependencies within F.

Theorem 2. Let Z, $\mathbf{M}_{ZZ}$, $\mathbf{A}$, $\mathbf{M}_0$, and F_+ be as in Theorem 1, i.e. $\mathbf{M}_{ZZ} = \mathbf{AM}_0\mathbf{A}'$ and $F_+ =_{\text{def}} \mathbf{A}^+Z$. Theorem 1 has shown that if $\mathbf{A}$ is L-invertible, F_+ is a solution and moreover the only solution for F_i in $Z = \mathbf{A}F_i$. But if $\mathbf{A}$ is L-ambiguous, then there are many m-tuples F_i of variables that $\mathbf{A}$-factor Z. Specifically, for any fixed $\mathbf{A}$ of rank h and column-order m in $\mathbf{M}_{ZZ} = \mathbf{AM}_0\mathbf{A}'$, $Z = \mathbf{A}F_i$ just in case $\mathbf{P}_A F_i = F_+$ ($\mathbf{P}_A = \mathbf{A}^+\mathbf{A}$) or, equivalently, iff

$$\begin{aligned} F_i &= F_+ + \mathbf{Q}_A X_i \\ &= F_+ + \mathbf{S}_A Y_i \end{aligned} \qquad (\mathbf{S}_A\mathbf{S}'_A = \mathbf{Q}_A = \mathbf{I} - \mathbf{P}_A) \tag{4}$$

for some arbitrary m-tuple X_i or $(m-h)$-tuple Y_i of variables, with $\mathbf{S}_A\mathbf{S}'_A$ any basic-structure decomposition of $\mathbf{Q}_A$. If $h = m$, $\mathbf{Q}_A = \mathbf{0}$ and $\mathbf{S}_A$ is null. But if $h < m$, $\mathbf{S}_A Y_i$ can be made non-zero (so that $F_i \neq F_+$) by choosing Y_i and hence $\mathbf{S}_A Y_i$ to span any arbitrary space of variables whose dimensionality does not exceed $\mathbf{A}$'s degree of L-ambiguity.

Corollary 1. Let $\mathbf{S}_A$ be as above, noting that the column-order $m-h$ of $\mathbf{S}_A$ is the degree of $\mathbf{A}$'s L-ambiguity. Then a tuple F_i of variables $\mathbf{A}$-factors Z just in case, when F_i is partitioned between its projection $\dot{F}_{i(Z)}$ into Z-space and its residual $E_{F_i \cdot Z}$ orthogonal to Z, $\dot{F}_{i(Z)}$ $\mathbf{A}$-factors Z while $E_{F_i \cdot Z} = \mathbf{S}_A E_i$ for some $(m-h)$-tuple of not necessarily-non-zero variables E_i orthogonal to Z.

Corollary 2. Let $\mathbf{S}_A$ be as above. Then (*a*) a tuple of variables F_i is an $\mathbf{A}$-factor of Z in Z-space just in case $F_i = F_+ + \mathbf{S}_A Y_i$ for some arbitrary $(m-h)$-tuple Y_i of variables in Z-space or, equivalently, just in case $F_i = (\mathbf{I} + \mathbf{S}_A\mathbf{W}_i)F_+$ for some arbitrary $(m-h) \times m$ coefficient matrix $\mathbf{W}_i$. (*b*) More generally, $\mathbf{F}_i$ is an $\mathbf{A}$-factor of Z just in case it has composition

$$F_i = (\mathbf{I} + \mathbf{S}_A\mathbf{W}_i)F_+ + \mathbf{S}_A E_i \qquad (\mathbf{M}_{ZE_i} = \mathbf{0}), \tag{5}$$

for an arbitrary $(m-h) \times m$ $\mathbf{W}_i$ and an arbitrary $(m-h)$-tuple E_i of variables orthogonal to Z.

Corollary 3. When $\mathbf{M}_{ZZ} = \mathbf{AM}_0\mathbf{A}'$ for an L-ambiguous $\mathbf{A}$ of rank h, $\mathbf{M}_0$ is just one of many moment matrices $\mathbf{M}_i$ that satisfy $\mathbf{M}_{ZZ} = \mathbf{AM}_i\mathbf{A}'$ for this same $\mathbf{A}$. One is always $\mathbf{M}_{F_+F_+} = \mathbf{A}^+\mathbf{M}_{ZZ}\mathbf{A}^{+\prime} = \mathbf{P}_A\mathbf{M}_0\mathbf{P}_A'$. But more comprehensively, $\mathbf{M}_{ZZ} = \mathbf{AM}_i\mathbf{A}'$ just in case

$$\mathbf{M}_i = (\mathbf{I}+\mathbf{S}_A\mathbf{W}_i)\mathbf{P}_A\mathbf{M}_0\mathbf{P}_A'(\mathbf{I}+\mathbf{S}_A\mathbf{W}_i)' + \mathbf{S}_A\mathbf{M}_e\mathbf{S}_A', \tag{6}$$

where $\mathbf{S}_A$ is as above, $\mathbf{W}_i$ is an arbitrary $(m-h)\times m$ matrix, and $\mathbf{M}_e$ is an arbitrary Gramian matrix of order $m-h$.

Proof. Let $\mathbf{P}_A =_{\text{def}} \mathbf{A}^+\mathbf{A}$ be the projector into $\mathbf{A}$-row space, with $\mathbf{Q}_A = \mathbf{S}_A\mathbf{S}_A'$ any fixed basic-structure decomposition of $\mathbf{P}_A$'s complement, and note that $\mathbf{AP}_A = \mathbf{A}$ while $\mathbf{AQ}_A = \mathbf{0}$. For any F_i such that $Z = \mathbf{A}F_i$, including special case $F_i = F_+$, $F_+ = \mathbf{A}^+Z = \mathbf{A}^+\mathbf{A}F_i = \mathbf{P}_AF_i$ and $F_i = (\mathbf{P}_A+\mathbf{Q}_A)F_i = \mathbf{P}_AF_i + \mathbf{Q}_AF_i = F_+ + \mathbf{Q}_AX_i$ for $X_i = F_i$. And conversely, $\mathbf{P}_AF_i = F_+$ entails $\mathbf{A}F_i = \mathbf{AP}_AF_i = \mathbf{A}F_+ = Z$, while for any F_i having composition (4), $\mathbf{A}F_i = \mathbf{A}F_+ + \mathbf{AQ}_AX_i = \mathbf{A}F_+ + \mathbf{0} = Z$. The equivalence of $\mathbf{Q}_AX_i$ for some X_i to $\mathbf{S}_AY_i$ for some Y_i is shown by $\mathbf{Q}_AX_i = \mathbf{S}_AY_i$ for $Y_i = \mathbf{S}_A'X_i$ while $\mathbf{S}_AY_i = \mathbf{Q}_AX_i$ for $X_i = \mathbf{S}_AY_i$. Finally, note that since $\mathbf{S}_A'(\mathbf{S}_AY_i) = Y_i$, Y_i and $\mathbf{S}_AY_i$ span the same space. ■ Corollary 1 follows from the equivalence of (4) to

$$F_i = [F_+ + \mathbf{S}_A\dot{Y}_{i(Z)}] + [\mathbf{S}_AE_{Y_i\cdot Z}],$$

wherein the first bracketed component, which satisfies (4), is $\dot{F}_{i(Z)}$, and the second is $E_{F_i\cdot Z}$. From there, Corollary 2*a* follows by observing that an $(m-h)$-tuple Y_i of variables is in Z-space (so that $Y_i = \dot{Y}_{i(Z)}$ and $E_{Y_i\cdot Z} = 0$) just in case, since F_+ spans Z-space, it equals $\mathbf{W}_iF_+$ for some $(m-h)\times m$ coefficient matrix $\mathbf{W}_i$. Corollary 2*b* follows immediately in light of Corollary 1. And Corollary 3 is direct from (5) and Guttman's Lemma.

Theorem 2 and its corollaries exhaustively answer Factor Determinacy Question **A**. But more than that, they do so insightfully, affording strong leverage on how, for any distinguished but L-ambigous $\mathbf{A}$, additional model constraints may further limit the range of solution alternatives. In particular, they illuminate the yield of one constraint that historically has been of special interest, and another that by rights ought to be. Suppose that we have developed a Z-moment decomposition $\mathbf{M}_{ZZ} = \mathbf{AM}_0\mathbf{A}'$ wherein $\mathbf{A}$, through a pattern we would like to retain for further analysis or interpretation of these data, is L-ambiguous. Then there are many $\mathbf{M}_i$ besides $\mathbf{M}_0$ that satisfy $\mathbf{M}_{ZZ} = \mathbf{AM}_i\mathbf{A}'$ for this same $\mathbf{A}$; and if we are willing to accept ones that are singular, Corollary 2 shows that we can always require F_i in $\langle Z = \mathbf{A}F_i, \mathbf{M}_{F_iF_i} = \mathbf{M}_i\rangle$ to lie in Z-space while still retaining a multiplicity of choices not merely for F_i but also for $\mathbf{M}_i$. On the other hand, our theory of Z's causal origin or at least our solution methodology may impose factor-moment constraints that are satisfied by $\mathbf{M}_0$ but by few if any other $\mathbf{M}_i$ in (6). If $\mathbf{M}_{ZZ} = \mathbf{AM}_0\mathbf{A}'$ is the decomposition of $\mathbf{M}_{ZZ}$ we most prefer, we then want to know – or at least the prominence of Factor Determinacy Question **AM** in our past literature urges us to care – which options exist for F_i in $\langle Z = \mathbf{A}F_i, \mathbf{M}_{F_iF_i} = \mathbf{M}_0\rangle$. The answer is almost immediate from Corollary 1 of Theorem 2:

Theorem 3. Let the rank-r moment matrix $\mathbf{M}_{ZZ}$ of variables Z (in P) have decomposition $\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_0\mathbf{A}'$, with $\mathbf{M}_0$ Gramian of order m and rank s, and $\mathbf{A}$ L-ambiguous to degree $d > 0$. And let $\mathbf{Q}_A = \mathbf{S}_A\mathbf{S}'_A$ be some fixed basic-structure decomposition of $\mathbf{Q}_A$ $(= \mathbf{I} - \mathbf{A}^+\mathbf{A})$. Then there exists a unique $\mathbf{A}$-factor F_0 of Z in Z-space, and a unique Gramian matrix $\mathbf{M}_e$ of order d and rank $s-r$, such that an m-tuple F_i of variables is an $\mathbf{A}$-factor of Z with moments $\mathbf{M}_{F_iF_i} = \mathbf{M}_0$ (in P) just in case the projection of F_i into Z-space is F_0 while $F_i - F_0 = \mathbf{S}_A E_i$ for some d-tuple E_i of variables orthogonal to Z and having moment matrix $\mathbf{M}_{E_iE_i} = \mathbf{M}_e$. (How to construct F_0 and $\mathbf{M}_e$ from the givens is shown in the proof.)

Corollary 1. Let Z, $\mathbf{A}$, and $\mathbf{M}_0$ be as above with the cardinality of P no less than the rank of $\mathbf{M}_0$. Then there is at least one solution for F_i in $\langle Z = \mathbf{A}F_i, \mathbf{M}_{F_iF_i} = \mathbf{M}_0 \rangle$. If there is only one – which obtains just in case $\mathbf{M}_e = \mathbf{0}$ – it lies in Z-space though it is not generally F_+. If more than one exists, there are infinitely many (unless the cardinality of P is only $r+1$, in which case E_i is unique up to reflection) and none lies in Z-space.

Proof. For any $\mathbf{A}$-factor F_i of Z whose moment matrix is given to be $\mathbf{M}_{F_iF_i} = \mathbf{M}_0$, the projection of F_i into Z-space is $F_0 =_{\text{def}} \dot{F}_{i(Z)} = (\mathbf{M}_{F_iZ}\mathbf{M}^+_{ZZ})Z = (\mathbf{M}_0\mathbf{A}'\mathbf{M}^+_{ZZ})Z$ or, equivalently, $F_0 = \dot{F}_{i(F_+)} = (\mathbf{M}_{F_iF_+}\mathbf{M}^+_{F_+F_+})F_+ = (\mathbf{M}_0\mathbf{P}_A\mathbf{M}^+_{F_+F_+})F_+$, the same for all. And when F_i is analysed as $F_i = F_0 + \mathbf{S}_A E_i$ in accord with Corollary 2 of Theorem 2, $\mathbf{M}_0 = \mathbf{M}_{F_0F_0} + \mathbf{S}_A\mathbf{M}_{E_iE_i}\mathbf{S}'_A$ or $\mathbf{M}_e =_{\text{def}} \mathbf{M}_{E_iE_i} = \mathbf{S}'_A(\mathbf{M}_0 - \mathbf{M}_{F_0F_0})\mathbf{S}_A$, again the same for all. Conversely, any $F_i = F_0 + \mathbf{S}_A E_i$ for some E_i orthogonal to Z with $\mathbf{M}_{E_iE_i} = \mathbf{M}_e$ is an $\mathbf{A}$-factor of Z for which $\mathbf{M}_{F_iF_i} = \mathbf{M}_0$. That $\mathbf{M}_e$ is Gramian follows from the existence of at least one such F_i so long as the cardinality of P is no less than the rank of $\mathbf{M}_0$ (Guttman's Lemma); while even if the size-of-P condition is unsatisfied, we can always construct a sufficiently large P_1 and Z_1 to have $\mathbf{M}_{Z_1Z_1} = \mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_0\mathbf{A}'$ and $\mathbf{M}_e$ hence Gramian by Guttman's Lemma. The order of $\mathbf{M}_e$ evidently equals the degree of $\mathbf{A}$'s L-ambiguity because the latter is the column-order of $\mathbf{S}_A$. And the rank of $\mathbf{M}_e$ equals the dimensionality s of F_i-space less the dimensionality r of Z-space because the space spanned jointly by Z and E_i with E_i orthogonal to Z is also the space spanned by F_i (since $Z = \mathbf{A}F_i$ and $E_i = \mathbf{S}'_A(F_i - \dot{F}_{(Z)})$). ■ As for the Corollary, its first claim is simply Guttman's Lemma. And its second claim is obvious, since $\mathbf{M}_{E_iE_i} = \mathbf{M}_e = \mathbf{0}$ just in case all variables in E_i are zero. Finally, for any Gramian $\mathbf{M}_e \neq \mathbf{0}$, there are infinitely many ways to choose E_i with moments $\mathbf{M}_{E_iE_i} = \mathbf{M}_e$ unless the size of P admits only one dimension of variables orthogonal to Z, namely, when P's cardinality exceeds the rank of $\mathbf{M}_{ZZ}$ just by 1. In that case, all variables in E are collinearities fixed by $\mathbf{M}_e$ save for simultaneous reflection of all.

Beyond study of constraints on $\mathbf{M}_i$ in $\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_i\mathbf{A}'$ insufficient to specify $\mathbf{M}_i$ uniquely, there seems little more to say about the indeterminacy of Z's $\mathbf{A}$-factors for any fixed $\mathbf{A}$. But there remains a complementary indeterminacy in factor *pattern* which also merits scrutiny. We have seen that when $\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_0\mathbf{A}'$ for some distinguished but L-ambiguous $\mathbf{A}$, there are many $\mathbf{M}_i$ such that $\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_i\mathbf{A}'$, some of which may well be as attention-worthy as $\mathbf{M}_0$. In particular, some such $\mathbf{M}_i$ might be singular; and in that case any $\mathbf{A}$-factor F of Z for which $\mathbf{M}_{FF} = \mathbf{M}_i$ also factors Z by various $\mathbf{A}_j$

other than **A**. If our purpose at hand finds **A**-invertibility attractive it should interest us to know whether these alternatives for $\mathbf{A}_j$ include ones less L-ambiguous than **A**. More generally, in response to Factor Indeterminacy Question **F** (equivalently, **FM**), can anything worthwhile be said about the range of patterns by which F factors Z? The answer:

Theorem 4. Let F be an m-tuple of variables that factors variable n-tuple Z, with Z-space and F-space having respective dimensionalities r and s. That is, r is the rank of $\mathbf{M}_{ZZ}$ while s and m are respectively the rank and order of $\mathbf{M}_{FF}$ so that $r \leqslant s \leqslant m$. Then the ranks of the $n \times m$ coefficient matrices $\mathbf{A}_i$ by which F factors Z, i.e. for which $Z = \mathbf{A}_i F$, range over all integers in the interval from r to $r+m-s$ inclusive. Starting from a basic-structure decomposition $\mathbf{M}_{FF} = \mathbf{R}_0\mathbf{D}_0^2\mathbf{R}_0'$ of F's moment matrix and any **A** by which F **A**-factors Z, the exact range of $\mathbf{A}_i$ in $Z = \mathbf{A}_i F$ is

$$\mathbf{A}_i = \mathbf{A}\mathbf{P}_0 + \mathbf{W}_i\mathbf{S}_0' \qquad (\mathbf{P}_0 = \mathbf{R}_0\mathbf{R}_0',\ \mathbf{S}_0\mathbf{S}_0' = \mathbf{Q}_0 = \mathbf{I} - \mathbf{P}_0), \tag{7}$$

where $\mathbf{W}_i$ ranges over all $m \times (m-s)$ real matrices.

Corollary 1. If the F given above lies in Z-space, it **A**-factors Z by some L-invertible **A**. Put more strongly, $Z = \mathbf{A}F$ for an L-invertible **A** just in case F spans Z-space.

Corollary 2. The L-invertible **A** of Corollary 1, or more generally the $\mathbf{A}_i$ in $Z = \mathbf{A}_i F$ at any attainable rank, is not unique unless F is a basis for its space.

Proof. Let n-tuple Z lie in a possibly-proper subspace of the space spanned by m-tuple F, while F's moment matrix has basic-structure $\mathbf{M}_{FF} = \mathbf{R}_0\mathbf{D}_0^2\mathbf{R}_0'$. Then factor s-tuple $G =_{\text{def}} \mathbf{R}_0'F$ is a basis for F-space with moments $\mathbf{M}_{GG} = \mathbf{D}_0^2$ while

$$F = \mathbf{R}_0 G.$$

And since variables Z, too, lie in G-space without necessarily spanning it,

$$Z = \mathbf{B}_0 G$$

for some $n \times s$ coefficient matrix $\mathbf{B}_0$ whose rank is also the rank r of $\mathbf{M}_{ZZ}$. Now, for any $n \times m$ $\mathbf{A}_i$, F $\mathbf{A}_i$-factors Z iff $\mathbf{B}_0 G = Z = \mathbf{A}_i F = \mathbf{A}_i\mathbf{R}_0 G$, i.e. iff

$$\mathbf{B}_0 = \mathbf{A}_i\mathbf{R}_0, \tag{8}$$

since $\mathbf{B}_0 G = \mathbf{A}_i\mathbf{R}_0 G$ entails $\mathbf{B}_0\mathbf{M}_{GG} = \mathbf{A}_i\mathbf{R}_0\mathbf{M}_{GG}$ whose post-multiplication by $\mathbf{M}_{GG}^{-1}$ yields (8). Define

$$\mathbf{A}_0 =_{\text{def}} \mathbf{B}_0\mathbf{R}_0', \qquad \mathbf{P}_0 =_{\text{def}} \mathbf{R}_0\mathbf{R}_0', \qquad \mathbf{Q}_0 = \mathbf{I} - \mathbf{P}_0 = \mathbf{S}_0\mathbf{S}_0',$$

where $\mathbf{S}_0$ is any fixed orthonormal completion of $\mathbf{R}_0$ so that the order of $\mathbf{S}_0$ is $m \times (m-s)$ while $\mathbf{R}_0'\mathbf{S}_0 = \mathbf{0}$. Then (8) holds iff $\mathbf{B}_0\mathbf{R}_0' = \mathbf{A}_i\mathbf{R}_0\mathbf{R}_0'$, i.e. iff

$$\mathbf{A}_0 = \mathbf{A}_i\mathbf{P}_0. \tag{9}$$

And $\mathbf{A}_i$ satisfies (9) just in case $\mathbf{A}_i$ has composition

$$\mathbf{A}_i = \mathbf{A}_0 + \mathbf{W}_i\mathbf{S}_0' \tag{10}$$

for some $n \times (m-s)$ matrix $\mathbf{W}_i$. For (10) entails $\mathbf{A}_i\mathbf{P}_0 = \mathbf{A}_0\mathbf{P}_0 + \mathbf{W}_i\mathbf{S}_0'\mathbf{P}_0 = \mathbf{A}_0$ since $\mathbf{A}_0\mathbf{P}_0 = \mathbf{A}_0$ and $\mathbf{S}_0'\mathbf{P}_0 = \mathbf{0}$, while conversely, (9) entails $\mathbf{A}_i = \mathbf{A}_i(\mathbf{P}_0 + \mathbf{Q}_0) = \mathbf{A}_i\mathbf{P}_0 + \mathbf{A}_i\mathbf{Q}_0 = \mathbf{A}_0 + \mathbf{W}_i\mathbf{S}_0'$ for $\mathbf{W}_i = \mathbf{A}_i\mathbf{S}_0$. And (7) is immediate from (9) and (10). The solution for $\mathbf{A}_i$ in (10) is unique (at $\mathbf{A}_i = \mathbf{A}_0$ with rank r) just in case $m = s$, i.e. iff F is a basis for its space. Alternatively, suppose $s < m$. Then it remains to show that choice of $\mathbf{W}_i$ in (10) can put the rank h of $\mathbf{A}_i$ anywhere between r and $r+(m-s)$, inclusive. Since h is also the rank of $\mathbf{A}_i\mathbf{A}_i'$ (cf. the basic-structure of $\mathbf{A}_i$ and $\mathbf{A}_i\mathbf{A}_i'$) and is easier to analyse in the latter, write

$$\mathbf{A}_i\mathbf{A}_i' = (\mathbf{B}_0\mathbf{R}_0' + \mathbf{W}_i\mathbf{S}_0')(\mathbf{B}_0\mathbf{R}_0' + \mathbf{W}_i\mathbf{S}_0')' = \mathbf{B}_0\mathbf{B}_0' + \mathbf{W}_i\mathbf{W}_i'. \tag{11}$$

If we choose $\mathbf{W}_i = \mathbf{0}$, the rank of $\mathbf{A}_i$ evidently equals that of $\mathbf{B}_0$, namely r. Alternatively, if $\mathbf{W}_i \neq \mathbf{0}$, the right-hand matrix products in (11) have some basic-structure decompositions

$$\mathbf{B}_0\mathbf{B}_0' = \mathbf{R}_b\mathbf{D}_b^2\mathbf{R}_b', \qquad \mathbf{W}_i\mathbf{W}_i' = \mathbf{R}_w\mathbf{D}_w^2\mathbf{R}_w'$$

with the column-order of $\mathbf{R}_b$ equalling the rank r of $\mathbf{B}_0$ while the column-order of $\mathbf{R}_w$ is some positive integer k, set by choice of $\mathbf{W}_i$, no greater than the column-order $m-s$ of $\mathbf{W}_i$. Within these limits on k, we can make $\mathbf{R}_w$ and $\mathbf{D}_w \neq \mathbf{0}$ in the basic structure of $\mathbf{W}_i\mathbf{W}_i'$ anything we wish by taking $\mathbf{R}_w\mathbf{D}_w$ for $\mathbf{W}_i$. For any such choice, (11) continues as

$$\mathbf{A}_i\mathbf{A}_i' = \mathbf{R}_b\mathbf{D}_b^2\mathbf{R}_b' = [\mathbf{R}_b\ \mathbf{R}_w]\begin{bmatrix}\mathbf{D}_b^2 & \\ & \mathbf{D}_w^2\end{bmatrix}[\mathbf{R}_b\ \mathbf{R}_w]'. \tag{11'}$$

(*Note*. (11') still holds if $\mathbf{D}_w = \mathbf{0}$, but $\mathbf{R}_w\mathbf{D}_w^2\mathbf{R}_w'$ is then not a basic-structure decomposition of $\mathbf{W}_i\mathbf{W}_i'$ as stipulated.) It is clear from (11') that the rank of $\mathbf{A}_i\mathbf{A}_i'$ cannot exceed the rank of $\mathbf{D}_b$ plus the rank chosen for $\mathbf{D}_w$, so $h \leqslant (r+k) \leqslant (r+m-s)$ with $h = r$ if $\mathbf{D}_w^2$ is replaced by $\mathbf{0}$. But for any choice of positive k up to this limit, there exists an $n \times k$ rectinormal $\mathbf{S}_w$ that is orthogonal to $\mathbf{R}_b$, namely, the first k columns in any orthonormal completion of $\mathbf{R}_b$. And with this $\mathbf{S}_w$ taken for $\mathbf{R}_w$ together with any conforming choice of $\mathbf{D}_w$, the right-hand side of (11') becomes a basic-structure decomposition of $\mathbf{A}_i\mathbf{A}_i'$ with $r+k$ positive roots. That is, taking $\mathbf{W}_i = \mathbf{S}_w = \mathbf{S}_w\mathbf{D}_w$ for any rank-k $\mathbf{D}_w$ and k-columned rectinormal $\mathbf{S}_w$ orthogonal to $\mathbf{R}_w$ ($k \leqslant m-s$) yields an $\mathbf{A}_i$ in (10) having rank $h = (r+k) \leqslant (r+m-s)$. ■ Proof of corollaries: The first version of Corollary 1 follows by noting that when Z-space and F-space have the same dimensionality, $s = r$ so that the range of ranks attainable by $\mathbf{A}_i$ includes column-order m; it is strengthened into a biconditional by the entailment for L-invertible $\mathbf{A}_i$ from $Z = \mathbf{A}_iF$ to $F = \mathbf{A}_i^LZ$. And Corollary 2 is evident from (11') in that with $\mathbf{R}_w = \mathbf{S}_w$ at any attainable rank $h \geqslant r+1$ for $\mathbf{A}_i$, each different $\mathbf{D}_w$ yields a different $\mathbf{A}_i$. To include case $h = r$, take $\mathbf{R}_w = \mathbf{R}_b$ with any order-r $\mathbf{D}_w$.

Finally, with no great enthusiasm but for the sake of completeness, we observe

> *Theorem 5*. Given that the moment matrix of an n-tuple Z of variables has decomposition $\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_0\mathbf{A}'$ for a distinguished $m \times m$ $\mathbf{M}_0$, let
>
> $$\mathbf{M}_{ZZ} = \mathbf{R}_z\mathbf{D}_z^2\mathbf{R}_z', \qquad \mathbf{M}_0 = \mathbf{R}_0\mathbf{D}_0^2\mathbf{R}_0'$$
>
> be basic-structure decompositions of $\mathbf{M}_{ZZ}$ and $\mathbf{M}_0$ wherein the rank of $\mathbf{M}_{ZZ}$ and

hence column-order of $\mathbf{R}_z$ is r, while the rank of $\mathbf{M}_0$ and hence column-order of $\mathbf{R}_0$ is s, so that $r \leqslant s \leqslant m$. Also, let $\mathbf{S}_0$ be any $m \times (m-s)$ orthonormal completion of this $m \times s$ $\mathbf{R}_0$. Then any F_j that factors Z has moments $\mathbf{M}_{F_jF_j} = \mathbf{M}_0$ just in case

$$F_j = \mathbf{R}_0\mathbf{D}_0\mathbf{T}_jG_j \qquad (G_j = \langle G_z, G_e\rangle, G_z = \mathbf{D}_z^{-1}\mathbf{R}_z'Z) \tag{12}$$

for an arbitrary $m \times m$ orthonormal $\mathbf{T}_j$ and an arbitrary $(s-r)$-tuple G_e of orthonormal variables orthogonal to Z. And a coefficient matrix $\mathbf{A}_i$ satisfies $\mathbf{M}_{ZZ} = \mathbf{A}_i\mathbf{M}_0\mathbf{A}_i'$ just in case

$$\mathbf{A}_i = \mathbf{R}_z\mathbf{D}_z\mathbf{R}_j'\mathbf{D}_0^{-1}\mathbf{R}_0' + \mathbf{W}_i\mathbf{S}_0' \tag{13}$$

for an arbitrary $\mathbf{W}_i$ of order $n \times (m-s)$ and an arbitrary $m \times r$ rectinormal $\mathbf{R}_j$. We can choose $\mathbf{W}_i$ in (13) to put the rank h of $\mathbf{A}_i$ anywhere in the interval $r \leqslant h \leqslant r + (m-s)$.

Proof. An F_j with $\mathbf{M}_{F_jF_j}$ of rank s factors Z iff F_j spans some s-dimensional space that includes Z; while then $\mathbf{M}_{F_jF_j} = \mathbf{M}_0 = \mathbf{R}_0\mathbf{D}_0^2\mathbf{R}_0'$ just in case $F_j = \mathbf{R}_0\mathbf{D}_0\mathbf{T}_jG_j$ for some $s \times s$ orthonormal rotation $\mathbf{T}_j$ of any fixed orthonormal basis G_j of F_j-space. And for any such F_j, we can always take G_j to be $G_j = \langle G_z, G_e\rangle$, where $G_z = \mathbf{D}_z^{-1}\mathbf{R}_z'Z$ comprises the r variance-normalized principal axes of Z while G_e is an arbitrary orthonormal $(s-r)$-tuple of orthonormal variables orthogonal to Z. Since $Z = \mathbf{R}_z\mathbf{D}_zG_z = [\mathbf{R}_z\mathbf{D}_z\ 0]G_j$ while G_j can be recovered from F_j by $G_j = \mathbf{T}_j'\mathbf{D}_0^{-1}\mathbf{R}_0'F_j$, F_j $\mathbf{A}_i$-factors Z for, *inter alia*, $\mathbf{A}_i = [\mathbf{R}_z\mathbf{D}_z\ 0]\mathbf{T}_j'\mathbf{D}_0^{-1}\mathbf{R}_0' = \mathbf{R}_z\mathbf{D}_z\mathbf{R}_j'\mathbf{D}_0^{-1}\mathbf{R}_0'$ where $\mathbf{R}_j$ comprises the first r columns of $\mathbf{T}_j$ and is hence $m \times r$. So (13) follows directly from (7) in Theorem 4, as does the claim about $\mathbf{A}_i$'s rank.

The second component on the right in (13) is indeterminacy in $\mathbf{A}_i$, given $\mathbf{M}_0$, that accrues from singularity of $\mathbf{M}_0$ and vanishes if $s = r$. But the range of $\mathbf{A}_i$ due to arbitrary $\mathbf{R}_j$ cannot be ameliorated by special properties of $\mathbf{M}_0$. Unlike the other varieties of factor indeterminacy examined here, there are no non-degenerate conditions on the fixed solution-fragment in this case that shrink its indeterminacy to unique specification. Even so, before dismissing this Theorem as utterly useless, note that it generalizes a principle which has been basic for traditional factor extraction's fixation of initial-factor moments at $\mathbf{M}_0 = \mathbf{I}_r$. In this special case, G_e and $\mathbf{W}_i$ are null, $\mathbf{R}_0 = \mathbf{D}_0 = \mathbf{I}_r$, and $\mathbf{R}_j = \mathbf{T}_j$; whence (12, 13) describes the class of all orthonormal bases of Z-space, with $\mathbf{T}_j = \mathbf{I}_r$ picking out the normalized principal axes of Z.

5. Conclusions

So what do these results signify for multivariate practice? Directly, not much; but indirectly, perhaps more than meets the eye. Mainly, they promote abstract mathematical comprehension of factor-indeterminacy relations in some breadth and depth, which not only is its own intellectual reward but profers moorings for the theories of particular structured models yet to appear. (Or at least its findings on L-invertibility are needed to secure the rationale of quad-factoring; and who can say where it will help out next.) But more than that, it redirects concern for factor indeterminacy from its narrow and – let us be honest – inconsequential classic **AF** focus into a perspective far more consilient with recent multivariate advances.

Model fitting in practice is often an iterated whipsawing whereby a provisionally fixed estimate of one model fragment is used to anchor a provisional solution for another. Frequently, the anchor comprises the current approximation to $\mathbf{M}_{ZZ}$ for just a latent component Z of the variables Y on which we have sample scores, together with part of a form-(1) decomposition of $\langle Z, \mathbf{M}_{ZZ}\rangle$; and the immediate task is to find enough of the decomposition's remainder to get on with what comes next. (Usually this solution reproduces $\mathbf{M}_{ZZ}$ only as an approximation thereupon taken to update the latter. Classically, $\mathbf{M}_{ZZ}$ is $\mathbf{M}_{YY}$ expunged of uniqueness; but modern practice also has more elaborate ways to fractionate $\mathbf{M}_{YY}$ into additive components estimating moments within and between blocks of Y's latent sources.) However we arrive at this provisional $\mathbf{M}_{ZZ}$, we no longer need to decompose it first by solving for some $\mathbf{A}$ in $\langle \mathbf{M}_{ZZ} = \mathbf{AM}_0\mathbf{A}', Z = \mathbf{A}F, \mathbf{M}_{FF} = \mathbf{M}_0\rangle$ under anchoring factor moments $\mathbf{M}_0 = \mathbf{I}_m$ for m equal to the (reproduced) rank r of $\mathbf{M}_{ZZ}$, and only later search for an interesting pattern of Z on some other basis of Z-space. Instead, we can nowadays fit $\mathbf{A}$ and $\mathbf{M}_0$ jointly under constraints spread over both $\mathbf{A}$ and $\mathbf{M}_0$ without requiring $m = r$; and precisely because so many diverse allocations of such constraints are computationally feasible, our choices thereof at particular stages of model fitting need to be rationalized with some care. Especially important is to distinguish between constraints of convenience that select a determinate solution from a range of alternatives equally good for the purpose at hand and essential constraints that preserve anchors or optimize features we take to be distinctive of solutions most meaningful for interpretation.

Accordingly, for each type of solution-fragment such that $\mathbf{M}_{ZZ}$ conjoined with this part of $\mathbf{M}_{ZZ}$'s running form-(1) decomposition is likely to serve as anchor at some stage of fitting one or another style of structural model, it is clearly advantageous to have on record a computationally effective specification of the range of model-(1) indeterminacy given a solution-fragment of this type. To illustrate, suppose that we have reached a stage of model fitting at which our provisional estimates of $\mathbf{M}_{ZZ}$ and $\mathbf{A}$ are to anchor solution for the $\mathbf{M}_i$ we think best for the next reproduction of $\mathbf{M}_{ZZ}$ as $\mathbf{AM}_i\mathbf{A}'$. Then, disregarding complications due to imperfect model fit under a discretionary loss-function, we know from Theorem 2 precisely what our options are for $\mathbf{M}_i$: We are assured that $\mathbf{M}_i = \mathbf{A}^+\mathbf{M}_{ZZ}\mathbf{A}^{+\prime}$ is ideal if $\mathbf{A}$ is L-invertible; we can see that $\mathbf{M}_i = \mathbf{A}^+\mathbf{M}_{ZZ}\mathbf{A}^{+\prime}$ is also most convenient for an L-ambiguous $\mathbf{A}$ if it does not matter at this point which completion of $\mathbf{M}_{ZZ}$'s decomposition we select; and finally, if $\mathbf{A}$ is L-ambiguous but we have a computable criterion for discriminating better from worse among the solutions for $\mathbf{M}_i$ in $\mathbf{M}_{ZZ} = \mathbf{AM}_i\mathbf{A}'$, Theorem 2's Corollary 3 tells us how in principle to find the best one, namely, by non-linear programming applied to the criterion value of function (6)'s output over the range of free parameters $\langle \mathbf{W}_i, \mathbf{M}_e\rangle$ (or rather, over certain more computationally efficient equivalents to the latter.)

Of course, the particular model fragments studied here scarcely begin to cover all patterns of solution anchoring that can and probably will arise in practice. But Theorems 1–5 do give foundations and direction for whatever elaborations may prove to be wanted. Whether they are also relevant to current or imminent modelling practice depends largely on the extent to which, if at all, we shall be devising model structures wherein patterns on blocks of common factors are allowed to be L-ambiguous. I know of no specific cases where this has occurred (not even quad-factoring breaks that radically with tradition), and it would be foolish to abandon the security of Theorem 1 without good reason. Yet neither have we reason for confidence that Nature shares our abhorrence for L-ambiguity in multivariate causal dependencies; so we had best give thought to how we might detect this if it occurs.

Let me close with a last word – or at least what I hope is *my* last word – on classic factor indeterminacy. Unlike the other cases examined here, Variety **AM** has no relevance for modelling practice insomuch as we never have use for a determinate choice of factor scores at any stage of model fitting. So why, when we are given $\langle Z, \mathbf{M}_{ZZ}\rangle$ and have identified a distinguished **A** and $\mathbf{M}_0$ such that $\mathbf{M}_{ZZ} = \mathbf{A}\mathbf{M}_0\mathbf{A}'$, should we feel disturbed when L-ambiguity of **A** admits a multiplicity of F_i for which $\langle Z = \mathbf{A}F_i, \mathbf{M}_{F_iF_i} = \mathbf{M}_0\rangle$? If we simply wished to pick out a specific F_i in this solution-range without much caring which one we get, we could easily close out the indeterminacy by an arbitrary stipulation of E_i in (5) under Theorem 3's constraint $\mathbf{M}_{E_iE_i} = \mathbf{M}_\epsilon$. Whereas if some of these F_i seem more selection-worthy than others, it is again straightforward in principle to search out the optimal one if we can but devise some computable measure τ on score matrices in the **MF** solution-range such that $\tau(F_i)$ appraises the merit of selection F_i. (Or at least that search would be computationally feasible under an identified score matrix on Z as presumed by the Indeterminacy-**AM** literature).

I submit that the real problem here has little if anything to do with **AM**-indeterminacy of factors construed extensionally as number-valued functions on whatever population we take to be at issue. We *do* intuit that some score matrices in the **AM** solution-range are more meritorious than others, yet have little notion of how to distinguish them from their less worthy brethren by a computable τ on $\{F_i\}$. But such a τ would be of little use even if, contrary to all likelihood, we could operationalize it. For what we are seeking here is the factor solution specified without identification by some version of causal criterion (3) (p. 211, above). And what we want to learn is not so much F_i-scores in the **AM** solution-range most closely aligned with scores in P on causal sources of Z as the non-extensional nature of these causal variables – precisely what score matrices fail to tell us about the contrast-classes of properties on which they list numerical scale values. (If you did know scores in P on causal sources F of Z, but nothing else about F save statistics entailed by the $\langle Z, F\rangle$ distribution in P, what good would this information do you?)

In short, the feeling of unease occasioned by classic factor-indeterminacy is legitimate and indeed important. But its proper target of concern is not Variety-**AM** indeterminacy of factor scores but our flaccid conceptual grip on the logic of causality and the ontology of scientific variables.

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